Zeta Function Zeros, the Möbius Function, and Dirichlet Products

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Abstract

The convolution of the normalized differences of non-trivial Riemann zeta function zeros with the Möbius function appears to result in an ensemble of normal distributions. This hypothesis is tested using histograms, quantile-quantile plots, the Kolmogorov-Smirnov test, the Shapiro-Wilk test, and the Lilliefors test. The slopes and y-intercepts of the quantile-quantile plots are sufficiently close to the line $y = x$ that linear regression can also be done. As in testing of the Montgomery-Odlyzko Law, statistics of next-nearest neighbors are also investigated. The convolution of the differences of the eigenvalues of a Gaussian-random Hermitian matrix with the Möbius function also appears to result in an ensemble of normal distributions.

The convolution of the non-trivial zeta function zeros with the Möbius function appears to result in an ensemble of approximately linearly-increasing values. The mean of this distribution appears to increase more slowly than the Mertens function. The corresponding ensemble for the eigenvalues of a Gaussian-random Hermitian matrix is not linear (at least for matrix sizes less than 4500x4500). The mean of this distribution appears to approach zero.

A method for grouping the non-trival zeta function zeros into an ensemble of approximately linearly-increasing values, an ensemble of approximately quadratically-increasing values, an ensemble of approximately cubicly-increasing values, ..., etc. is given.

A method for estimating new non-trivial zeta function zeros (higher on the line $x = \frac{1}{2}$) from a database of existing zeros using Dirichlet products and Möbius inversion is given. The number of zeros that can be estimated at one time is at least as large as the database. The relative errors of the estimates can be made arbitrarily small by using a sufficiently large database. In this method, values are linearly interpolated between successive zeros. The Fourier transform of this modified Riemann spectrum still gives the prime powers. This modified Riemann spectrum can also be used to compute Riemann's function $R(x)$. The method is also applicable to the eigenvalues of a Gaussian-random Hermitian matrix.

1 Introduction

Odlyzko's [1] tables of the zeros of the Riemann zeta function will be used. All known non-trivial zeros of the zeta function are of the form $\rho_n = \frac{1}{2} + i\gamma_n$, $\gamma_n \in R$. The normalized differences as given by Odlyzko [2] are $\delta_n = (\gamma_{n+1} - \gamma_n)\frac{\log(\gamma_n/(2\pi))}{2\pi}$. δ_n have mean value 1 in the sense that for any positive integers N and M, $\sum_{n=N+1}^{N+M} \delta_n = M + O(\log NM)$. Let z_n denote $\sum_{i|n} \delta_i \mu(i)$ where μ is the Möbius function. The z_n values are grouped according to the prime factorizations of n. Let p, q, r, s, t, and u denote distinct primes (this number of distinct primes is sufficient for $n \leq 2,000,000$). The simplest grouping of z_n values is at n values of the form pq. (The grouping of z_n values at prime n values is not normally distributed.) The cumulative distribution function is denoted by $\Phi(p)$ where p are probabilities between 0 and 1. The probit function is the inverse cumulative distribution function for the standard normal distribution. Wichura's [3] algorithm (accurate to 16 decimal places) is used to compute the probit function. Wichura's criteria for a satisfactory implementation are met, that is, $Z_{.25} = -0.6744897501960817$, $Z_{.001} = -3.090232306167814$, and $Z_{10^{-20}} = -9.262340089798408$. The more general function $F^{-1}(p) = \mu + \sigma\Phi^{-1}(p)$ where μ and σ are the mean and standard deviation of the z_n distribution will be used to generate quantile-quantile plots. The Matlab command "normrnd" is used to generate random variates for the normal probability distribution.

2 The Ensemble of Normal Probability Distributions

For $n \leq 1,000,000$, there are 55 groups in the ensemble with a significantly large sample size. For a normal probability fit of the 209,867 z_n values for $n \leq 1,000,000$ and at n values of the form pq, the mean is -0.0480 with a 95% confidence interval of (-0.0507, -0.0452) and the standard deviation is 0.6372 with a 95% confidence interval of (0.6352, 0.6391). See Figure (1) for a plot of these sorted z_n values. The corresponding F^{-1} values are also shown in the plot. (The F^{-1} values are overlain with the sorted z_x values.) See Figure (2) for a quantile-quantile plot of the sorted z_n values and the F^{-1} values. For a linear least-squares fit of the curve, the slope is 1 with a 95% confidence interval of (0.9999, 1), the y-intercept is $-4.923 \cdot 10^{-5}$ with a 95% confidence interval of ($-7.316 \cdot 10^{-5}$, $-2.529 \cdot 10^{-5}$), SSE (sum of squared error) equals 6.532, R-squared (goodness-of-fit measure for linear regression) equals 0.9999, and RMSE (root mean square error) equals 0.005579. There are significant deviations from the reference line (the linear least-squares fit) in the tails of the quantile-quantile plot. See Figure (3) for a histogram (with 100 bins) of these values overlain with a histogram of randomly generated values for a normal probability distribution with the same mean and standard deviation. (The values have been scaled by a factor of 10.0 and offset by 50 bins to the right.) The histograms are almost the same. When the parameters are estimated from a sample, the Lilliefors test should be used. The null hypothesis is that the function is normal with unspecified mean and variance. At a significance level of 0.05, the null hypothesis is rejected by the Lilliefors test. The P-value (calculated probability of finding the observed results when the null hypothesis is true) is NaN (not-a-number), the test statistic is 0.0026 and the critical value is 0.0019. The test returns the approximate P-value via interpolation into the Lilliefors simulation table. A NaN is returned when P is not found within the interval $0.01 \leq p \leq .20$. These missing observations in the sample are ignored. A small P-value (typically less than or equal to 0.05) indicates strong evidence against the null hypothesis. A large P-value (greater than 0.05) indicates weak evidence against the null hypothesis. Sample values are standardized for the Kolmogorov-Smirnov test by subtracting the mean and then dividing by the standard deviation. Standardizing the samples is equivalent to setting the mean and variance of the reference distribution equal to that of the sample estimate. This changes the null distribution of the test statistic. Empirical evidence collected for the distributions considered here indicates that the test statistic of the Lilliefors test is approximately equal to that of the Kolmogorov-Smirnov test and that the critical value of the Kolmogorov-Smirnov test is about 1.52 times as large as that of the Lilliefors test. At a significance level of 0.05, the null hypothesis that the data comes from a normal distribution is rejected by the Kolmogorov-Smirnov test. The test statistic is 0.0026 and the critical value is 0.0020 (0.0030/1.52). The Kolmogorov-Smirnov statistic requires a relatively large number of data points to reject the null hypothesis. If the sample size is sufficiently large, the test may detect even "trivial" departures from the null hypothesis. See Figure (4) for a plot of the empirical CDF (cumulative distribution function) and the standard normal CDF. There is no discernible difference.

For a normal probability fit of the 168 z_n values for $n \leq 1,000,000$ and at n values of the form p^2, the mean is -0.0816 with a 95% confidence interval of (-0.1388, -0.0244) and the standard deviation is 0.3755 with a 95% confidence interval of (0.3392, 0.4206). See Figure (5) for a quantile-quantile plot of these sorted z_n values and the corresponding F^{-1} values. For a linear least-squares fit of the curve, the slope is 0.9603 with a 95% confidence interval of (0.9329, 0.9876) and the y-intercept is -0.003212 with a 95% confidence interval of (-0.01369, -0.007267), SSE=0.75, R-squared=0.9666, and RMSE=0.06721. At a significance level of 0.05, the null hypothesis is rejected by the Lilliefors test. The P-value is NaN, the test statistic is 0.0935, and the critical value is 0.0684. At a significance level of 0.05, the null hypothesis is rejected by the Kolmogorov-Smirnoff test. The test statistic is 0.0935, and the adjusted critical value (0.1037/1.52) is 0.0682. See Figure (6) for a plot of the empirical CDF and the standard normal CDF. Even for a sample this small, the CDF's match fairly well.

For a normal probability fit of the 43,864 z_n values for $n \leq 1,000,000$ and at n values of the form p^2q, the mean is -0.0239 with a 95% confidence interval of $(-0.0296, -0.0181)$ and the standard deviation is 0.6103 with a 95% confidence interval of (0.6063, 0.6143). See Figure (7) for a quantile-quantile plot of these sorted z_n values and the corresponding F^{-1} values. For a linear least-squares fit of the curve, the slope is 0.9998 with a 95% confidence interval of (0.9997, 0.9999) and the y-intercept is $-4.727 \cdot 10^{-5}$ with a 95% confidence interval of $(-9.973 \cdot 10^{-5}, -5.187 \cdot 10^{-6})$, SSE=1.376, R-squared=0.9999, and RMSE=0.005601. See Figure (8) for a histogram (with 100 bins) of these values overlain with a histogram of randomly generated values for a normal probability distribution with the same mean and standard deviation. (The values have been scaled by a factor of 8.0 and offset by 50 bins to the right.) The histograms are almost the same (other than the peak of the z_n histogram being slightly shorter). At a significance level of 0.05, the null hypothesis is not rejected by the Lilliefors test. The P-value is 0.0780, the test statistic is 0.0040, and the critical value is 0.0042.

For a normal probability fit of the 17,459 z_n values for $n \leq 1,000,000$ and at n values of the form p^3q, the mean is 0.0145 with a 95% confidence interval of (0.0056, 0.0234) and the standard deviation is 0.6002 with a 95% confidence interval of (0.5940, 0.6066). See Figure (9) for a quantile-quantile plot of these sorted z_n values and the corresponding F^{-1} values. For a linear least-squares fit of the curve, the slope is 0.9994 with a 95% confidence interval of (0.9992, 0.9996), the y-intercept is $-8.338 \cdot 10^{-6}$ with a 95% confidence interval of $(-0.000138, 0.0001214)$, SSE=1.334, R-squared=0.9998, and RMSE=0.00874. There are significant deviations from the reference line in the tails of the quantile-quantile plot. See Figure (10) for a histogram (with 100 bins) of these values overlain with a histogram of randomly generated values for a normal probability distribution with the same mean and standard deviation. (The values have been scaled by a factor of 5.0 and offset by 50 bins to the right.) The histograms are almost the same. At a significance level of 0.05, the null hypothesis is rejected (just barely) by the Lilliefors test. The P-value is 0.0378, the test statistic is 0.0070, and the critical value is 0.0067.

For a normal probability fit of the 206,964 z_n values for $n \leq 1,000,000$ and at n values of the form pqr, the mean is 0.0158 with a 95% confidence interval of (0.0117, 0.0199) and the standard deviation is 0.9521 with a 95% confidence interval of (0.9492, 0.9550). See Figure (11) for a quantile-quantile plot of these sorted z_n values and the corresponding F^{-1} values. For a linear least-squares fit of the curve, the slope is 0.9999 with a 95% confidence interval of (0.9999, 0.9999), the y-intercept is $1.321 \cdot 10^{-5}$ with a 95% confidence interval of $(-1.224 \cdot 10^{-5}, 3.886 \cdot 10^{-5})$, SSE=0.0460, R-squared=1, and RMSE=0.005906. See Figure (12) for a histogram (with 100 bins) of these values overlain with a histogram of randomly generated values for a normal probability distribution with the same mean and standard deviation. (The values have been scaled by a factor of 9.0 and offset by 50 bins to the right.) The histograms are almost the same. At a significance level of 0.05, the null hypothesis is rejected (just barely) by the Lilliefors test. The P-value is 0.0460, the test statistic is 0.0020, and the critical value is 0.0019.

Similar results have been obtained for z_n values at n values of the forms p^4q, p^2q^2, p^5q, p^3q^2, p^2qr, p^6q, p^4q^2, p^3qr, $pqrs$, p^7q, p^8q, p^2q^2r, p^4qr, p^9q, p^3q^2r, p^2qrs, p^5qr, $p^{10}q$, $p^{11}q$, p^7qr, p^6qr, $p^2q^2r^2$, p^4q^2r, p^3q^3r, p^3qrs, $pqrst$, p^4qrs, p^2q^2rs, p^5q^2r, $p^3q^2r^2$, p^8qr, p^9qr, p^3q^2rs, p^2qrst, p^4q^3r, p^6q^2r, p^5qrs, p^5q^3r, p^7q^2r, p^4q^2rs, p^3qrst, p^6q^3r, p^6qrs, $pqrstu$, $p^2q^2r^2s$, p^3q^3rs, p^7qrs, p^5q^2rs, $p^3q^2r^2s$, and p^2q^2rst. The respective test statistics and critical values (at a significance level of 0.05) are (0.0074, 0.0098), (0.0305, 0.0522), (0.0117, 0.0140), (0.0399, 0.0631), (0.0024, 0.0030), (0.0155, 0.0195), (0.0614, 0.0853), (0.0044, 0.0049), (0.0051, 0.0029), (0.0242, 0.0270), (0.0405, 0.0370), (0.0076, 0.0095), (0.0071, 0.0073), (0.0556, 0.0503), (0.0138, 0.0113), (0.0064, 0.0037), (0.0087, 0.0107), (0.0728, 0.0686), (0.0968, 0.0914), (0.0134, 0.0220), (0.0103, 0.0154), (0.0668, 0.0763), (0.0195, 0.0169), (0.0256, 0.0276), (0.0056, 0.0063), (0.0073, 0.0065), (0.0072, 0.0099), (0.0106, 0.0087), (0.0180, 0.0244), (0.0952, 0.0719), (0.0164, 0.0314), (0.0189, 0.0447), (0.0117, 0.0109), (0.0100, 0.0078), (0.0345, 0.0301), (0.0287, 0.0345), (0.0085, 0.0153), (0.0325, 0.0443), (0.0334, 0.0481), (0.0184, 0.0174), (0.0102, 0.0144), (0.0559, 0.0636), (0.0112, 0.0231), (0.0196, 0.0252), (0.0273, 0.0339), (0.0141, 0.0286), (0.0290, 0.0352), (0.0314, 0.0269), (0.0385, 0.0356), and (0.0204, 0.0167). The respective slopes of the linear least-squares fits (of the quantile-quantile plots) are 0.9989, 0.9836, 0.9981, 0.9757, 0.9998, 0.9964, 0.9646, 0.9997, 0.9998, 0.9942, 0.9895, 0.9988, 0.9993, 0.9804, 0.9984, 0.9996, 0.9987, 0.9686, 0.948, 0.9996, 0.9978, 0.9623, 0.9971, 0.9935, 0.9993, 0.9992, 0.9988, 0.999, 0.9951, 0.9683, 0.9929, 0.9874, 0.9982, 0.9992, 0.9914, 0.9911, 0.9975, 0.9874, 0.9852, 0.9964, 0.9998, 0.9767, 0.9955, 0.9946, 0.9899, 0.9935, 0.9898, 0.9927, 0.9892, and 0.9971. The slopes are usually relatively small (less than 0.99) for small sample sizes. The respective y-intercepts of the linear least-square fits are $3.287 \cdot 10^{-5}$, -0.000383, 0.0001179, 0.0004057, $-2.202 \cdot 10^{-5}$, 0.0004119, 0.002177, $2.8 \cdot 10^{-5}$, $3.156 \cdot 10^{-5}$, 0.0006472, 0.001361, $-2.704 \cdot 10^{-5}$, $6.111 \cdot 10^{-5}$, 0.002654, $1.086 \cdot 10^{-5}$, $3.495 \cdot 10^{-5}$, 0.000229, 0.005501, 0.009329, 0.0006415, 0.0003692, 0.002545, $6.304 \cdot 10^{-5}$, -0.0003208, 0.0001188, 0.0001706, 0.0002333, $6.16 \cdot 10^{-5}$, 0.0001465, 0.004169, 0.001378, 0.001982, 0.0001907, $9.476 \cdot 10^{-5}$, -0.0004179, 0.0004685, 0.0005859, -0.0002908, 0.001071, 0.0005006, 0.0003821, 0.0002159, 0.0009297, -0.0003097, 0.0006202, 0.0007539, 0.002325, 0.001273, 0.00082, and 0.0001185. The absolute values of the y-intercepts are usually relatively large for small sample sizes. The respective R-squared values are 0.9997, 0.9974, 0.9994, 0.9921, 1, 0.9988, 0.994, 0.9999, 0.9997, 0.9985, 0.9963, 0.9994, 0.9997, 0.9893, 0.9992, 0.9996, 0.9995, 0.9837, 0.9678, 0.9994, 0.9996, 0.9791, 0.9989, 0.9977, 0.9995, 0.9993, 0.9996, 0.9994, 0.9989, 0.9864, 0.999, 0.9984, 0.9986, 0.9996, 0.9952, 0.9977, 0.999, 0.9981, 0.9968, 0.9978, 0.9995, 0.9949, 0.999, 0.9985, 0.9948, 0.9984, 0.9957, 0.9957, 0.9948, and 0.9987. The respective sample sizes are 8,115, 288, 4,017, 197, 87,338, 2,058, 108, 33,201, 92,966, 1,077, 574, 8,629, 14,676, 310, 6,185, 55,930, 6,883, 167, 94, 1,619, 3,312, 135, 2,735, 1,028, 19,538, 18,387, 7,930, 10,282, 1,318, 152, 797, 393, 6,648, 12,753, 867, 661, 3,367, 400, 340, 2,585, 3,807, 194, 1,465, 1,235, 683, 963, 633, 1,087, 618, and 2,825. This accounts for about 91.0% of the z_n values for n less than or equal to 1,000,000. Presumably, similar results will be obtained (when enough data is available) for all other curves except z_n values at n values of the form p.

The groups where the null hypothesis is rejected are at n values of the forms pq, p^2, p^3q, pqr, $pqrs$, p^8q, p^9q, p^3q^2r, p^2qrs, $p^{10}q$, $p^{11}q$, p^4q^2r, $pqrst$, p^2q^2rs, $p^3q^2r^2$, p^3q^2rs, p^2qrst, p^4q^3r, p^4q^2rs, p^5q^2rs, $p^3q^2r^2s$, and p^2q^2rst. The respective sample sizes are 209,867, 168, 17,459, 206,964, 92,966, 574, 310, 6,185, 55,930, 167, 94, 2,735, 18,387, 10,282, 152, 6,648, 12,753, 867, 2,585, 1,087, 618, and 2,825. This accounts for about 65.0% of the z_n values less than or equal to 1,000,000. (The ratio of the number of z_n values that failed the Lilliefors test to those that passed is about 2.496 (649,623/260,269)). Excluding the groups that almost passed (at n values of the forms p^3q and pqr) reduces this to about 42.5%. For the groups with a sample size less than 5,000, the null hypothesis is also rejected by the Shapiro-Wilk normality test. In addition, the Shapiro-Wilk test rejects the null hypothesis for the group at n values of the form $p^2q^2r^2s$ (consisting of 683 elements).

The $\delta_n + \delta_{n+1}$ values approach their asymptotic behavior more slowly than the δ_n values. In the following, adjacent z_n values are added after they are grouped. Groups that pass (but previously failed) the Lilliefors test are at n values of the forms p^3q, pqr, p^8q, p^9q, p^2qrs, $p^{10}q$, $p^{11}q$, p^4q^2r, $pqrst$, p^2q^2rs, $p^3q^2r^2$, p^2qrst, p^4q^3r, p^4q^2rs, p^5q^2rs, $p^3q^2r^2s$, and p^2q^2rst. The respective sample sizes are 17,458, 206,963, 573, 309, 55,929, 167, 93, 2,734, 18,386, 10,281, 151, 12,752, 866, 2,584, 1,086, 617, and 2,824. This accounts for about 33.4% of the z_n values less than or equal to 1,000,000. Groups that fail (but previously passed) the Lilliefors test are at n values of the forms p^3qrs, p^6qrs, and p^7qrs. The respective sample sizes are 19,537, 1,464, and 632. This accounts for about 2.2% of the z_n values less than or equal to 1,000,000.

For a normal probability fit of the 8,114 $z_n + z_{n+1}$ values for $n \leq 1,000,000$ and at n values of the form p^4q, the mean is 0.1025 with a 95% confidence interval of (0.0846, 0.1204) and the standard deviation is 0.8221 with a 95% confidence interval of (0.8096, 0.8349). See Figure (13) for a quantile-quantile plot of these sorted $z_n + z_{n+1}$ values and the corresponding F^{-1} values. For a linear least-squares fit of the curve, the slope is 0.999 with a 95% confidence interval of (0.9986, 0.9993), the y-intercept is 0.0001363 with a 95% confidence interval of (-0.0001379, 0.00041061), SSE=1.269, R-squared=0.9998, and RMSE=0.01251. At a significance level of 0.05, the null hypothesis is not rejected by the Lilliefors test. The P-value is NaN, the test statistic is 0.0076, and the critical value is 0.0098.

For a normal probability fit of the 287 $z_n + z_{n+1}$ values for $n \leq 1,000,000$ and at n values of the form p^2q^2, the mean is -0.0478 with a 95% confidence interval of (-0.1357, 0.0401) and the standard deviation is 0.7565 with a 95% confidence interval of (0.6992, 0.8240). See Figure (14) for a quantile-quantile plot of these sorted $z_n + z_{n+1}$ values and the corresponding F^{-1} values. For a linear least-squares fit of the curve, the slope is 0.9819 with a 95% confidence interval of (0.973, 0.9908), the y-intercept is -0.0008932 with a 95% confidence interval of (-0.007617, 0.005831), SSE=0.9507, R-squared=0.994, and RMSE=0.05776. At a significance level of 0.05, the null hypothesis is not rejected by the Lilliefors test. The P-value is NaN, the test statistic is 0.0285, and the critical value is 0.0523.

For a normal probability fit of the 4,016 $z_n + z_{n+1}$ values for $n \leq 1,000,000$ and at n values of the form $p^5 q$, the mean is 0.1644 with a 95% confidence interval of (0.1392, 0.1896) and the standard deviation is 0.8146 with a 95% confidence interval of (0.7972, 0.8328). See Figure (15) for a quantile-quantile plot of these sorted $z_n + z_{n+1}$ values and the corresponding F^{-1} values. For a linear least-squares fit of the curve, the slope is 0.9982 with a 95% confidence interval of (0.9977, 0.9987), the y-intercept is 0.0002639 with a 95% confidence interval of (-0.0001542, 0.0006821), SSE=0.7046, R-squared=0.9997, and RMSE=0.01325. At a significance level of 0.05, the null hypothesis is not rejected by the Lilliefors test. The P-value is NaN, the test statistic is 0.0075, and the critical value is 0.0140.

For a normal probability fit of the 196 $z_n + z_{n+1}$ values for $n \leq 1,000,000$ and at n values of the form $p^3 q^2$, the mean is 0.0417 with a 95% confidence interval of (-0.0661, 0.1495) and the standard deviation is 0.7650 with a 95% confidence interval of (0.6960, 0.8493). See Figure (16) for a quantile-quantile plot of these sorted $z_n + z_{n+1}$ values and the corresponding F^{-1} values. For a linear least-squares fit of the curve, the slope is 0.9759 with a 95% confidence interval of (0.9642, 0.9876), the y-intercept is 0.0009936 with a 95% confidence interval of (-0.007951, 0.009938), SSE=0.7798, R-squared=0.9929, and RMSE=0.0634. At a significance level of 0.05, the null hypothesis is not rejected by the Lilliefors test. The P-value is NaN, the test statistic is 0.0511, and the critical value is 0.0633.

For a normal probability fit of the 87,337 $z_n + z_{n+1}$ values for $n \leq 1,000,000$ and at n values of the form $p^2 qr$, the mean is 0.0886 with a 95% confidence interval of (0.0801, 0.0972) and the standard deviation is 1.2953 with a 95% confidence interval of (1.2893, 1.3014). See Figure (17) for a quantile-quantile plot of these sorted $z_n + z_{n+1}$ values and the corresponding F^{-1} values. For a linear least-squares fit of the curve, the slope is 0.9998 with a 95% confidence interval of (0.9998, 0.9999), the y-intercept is $-3.539 \cdot 10^{-5}$ with a 95% confidence interval of ($-8.266 \cdot 10^{-5}$, $1.188 \cdot 10^{-5}$), SSE=4.416, R-squared=1, and RMSE=0.007111. At a significance level of 0.05, the null hypothesis is not rejected by the Lilliefors test. The P-value is NaN, the test statistic is 0.0022, and the critical value is 0.0030.

3 The Ensemble of Normal Probability Distributions for Zeros Higher on the Line $x = 1/2$

A difficulty with computing z_n values for zeta function zeros higher on the line $x = 1/2$ is that all the zeros up to that height must be available. For $1,000,001 \leq n \leq 2,000,000$, the null hypothesis is rejected by the Lilliefors test for groups at n values of the forms pq, $pqrs$, p^2q^2r, p^2qrs, p^3qrs, $pqrst$, p^5q^3r, p^3qrst, and $p^3q^2r^2s$. The respective test statistics and critical values are (0.0034, 0.0020), (0.0032, 0.0027), (0.0104, 0.0104), (0.0049, 0.0036), (0.0073, 0.0060), (0.0063, 0.0057), (0.0505, 0.0487), (0.0137, 0.0121), and (0.0429, 0.0364). See Figures (18) through (26) for quantile-quantile plots of these groups. The respective slopes of the reference lines are 0.9999, 0.9998, 0.9998, 0.9997, 0.9995, 0.9995, 0.9822, 0.9983, and 0.9879. The respective y-intercepts are $-3.33 \cdot 10^{-5}$, $8.459 \cdot 10^{-6}$, $5.28 \cdot 10^{-5}$, $3.217 \cdot 10^{-6}$, $3.84 \cdot 10^{-5}$, 0.0001036, -0.0006766, 0.0002712, and 0.0002291. The respective R-squared values are 0.9999, 0.9998, 0.9996, 0.9997, 0.9997, 0.9997, 0.9915, 0.9991, and 0.9927. The respective sample sizes are 197,194, 104,322, 7,322, 60,656, 21,543, 24,568, 331, 5,398, and 593. This accounts for about 42.2% of the z_n values between 1,000,001 and 2,000,000. Excluding the group that almost passed (at n values of the form p^2q^2r) reduces this to about 41.5%. The ratio of the number of z_n values that failed the Lilliefors test to those that passed is about 0.8756 (422,065/482,018). This is a substantial improvement over the 65.0% that were rejected for z_n values less than or equal to 1,000,000. For $1,000,001 \leq n \leq 2,000,000$, about 90.4% of the z_n values are in the ensemble. This is slightly less than the percentage for $n \leq 1,000,000$, so a better measure of the improvement is the fail/pass ratio (2.496 for $n \leq 1,000,000$).

For $2,000,001 \leq n \leq 3,000,000$, the null hypothesis is rejected by the Lilliefors test for groups at n values of the forms pq, p^2q, $pqrs$, p^2qrs, p^3qrs, p^2q^2rs, p^2qrst, and p^7q^2r. The respective test statistics and critical values are (0.0034, 0.0020), (0.0050, 0.0046), (0.0048, 0.0027), (0.0048, 0.0036), (0.0069, 0.0060), (0.0092, 0.0087), (0.0068, 0.0065), and (0.0560, 0.0549). See Figures (27) through (34) for quantile-quantile plots of these groups. The respective slopes of the reference lines are 0.9998, 0.9997, 0.9998, 0.9997, 0.9994, 0.999, 0.9993, and 0.9748. The respective y-intercepts are $-5.486 \cdot 10^{-5}$, $2.171 \cdot 10^{-5}$, $4.304 \cdot 10^{-5}$, $6.58 \cdot 10^{-5}$, 0.0001029, 0.0001181, $9.935 \cdot 10^{-5}$, and 0.0004454. The respective R-squared values are 0.9999, 0.9999, 0.9998, 0.9997, 0.9997, 0.9995, 0.9995, and 0.9825. The respective sample sizes are 193,116, 36,632, 108,253, 61,819, 22,100, 10,485, 18,761, and 260. This accounts for about 45.1% of the z_n values between 2,000,001 and 3,000,000. Excluding the group that almost passed (at n values of the form p^2qrst) reduces this to about 43.3%. The ratio of the number of z_n values that failed the Lilliefors test to those that passed is about 0.976 (451,426/462,489). For $2,000,001 \leq n \leq 3,000,000$, about 91.4% of the z_n values are in the ensemble.

For $3,000,001 \le n \le 4,000,000$, the null hypothesis is rejected by the Lilliefors test for groups at n values of the forms pq, p^2, p^3q, pqr, p^4q, p^4q^2, $pqrs$, p^7q, p^2q^2r, p^3q^2r, p^2qrs, $pqrst$, and p^3q^2rs. The respective test statistics and critical values are (0.0044, 0.0020), (0.2009, 0.1519), (0.0075, 0.0075), (0.0022, 0.0019), (0.0111, 0.0110), (0.1938, 0.1832), (0.0032, 0.0027), (0.0315, 0.0309), (0.0116, 0.0108), (0.0144, 0.0129), (0.0046, 0.0035), (0.0060, 0.0053), and (0.0128, 0.0107). See Figures (35) through (47) for quantile-quantile plots of these groups. The respective slopes of the reference lines are 0.9998, 0.8576, 0.9992, 1, 0.9985, 0.8731, 0.9999, 0.9914, 0.9987, 0.9981, 0.9997, 0.9996, and 0.9985. The respective y-intercepts are $-1.64 \cdot 10^{-5}$, -0.01312, $3.608 \cdot 10^{-5}$, $-3.154 \cdot 10^{-5}$, $4.868 \cdot 10^{-5}$, -0.01215, $-1.583 \cdot 10^{-5}$, 0.001011, $5.006 \cdot 10^{-5}$, $6.088 \cdot 10^{-5}$, $4.512 \cdot 10^{-5}$, $3.828 \cdot 10^{-5}$, and 0.0001901. The respective R-squared values are 0.9998, 0.8613, 0.9997, 1, 0.9992, 0.9454, 0.9999, 0.9956, 0.9995, 0.9992, 0.9997, 0.9997, and 0.9992. The respective sample sizes are 190,506, 34, 14,110, 209,072, 6,499, 22, 110,409, 824, 6,695, 4,753, 62,558, 28,451, and 6,913. This accounts for about 64.1% of the z_n values between 3,000,001 and 4,000,000. Excluding the groups that almost passed (at n values of the forms p^3q, pqr, and p^4q) reduces this to about 41.1%. There is little deviation from the reference lines for the groups at n values of the forms pqr and $pqrs$. The ratio of the number of z_n values that failed the Lilliefors test to those that passed is about 2.345 (640,846/273,283). For $3,000,001 \le n \le 4,000,000$, about 91.4% of the z_n values are in the ensemble.

For $4,000,001 \le n \le 5,000,000$, the null hypothesis is rejected by the Lilliefors test for groups at n values of the forms pq, $pqrs$, p^4qr, p^2qrs, p^3qrs, $pqrst$, and p^2q^2rs. The respective test statistics and critical values are (0.0039, 0.0020), (0.0041, 0.0026), (0.0090, 0.0076), (0.0046, 0.0035), (0.0061, 0.0059), (0.0092, 0.0051), and (0.0089, 0.0087). See Figures (48) through (54) for quantile-quantile plots of these groups. The respective slopes of the reference lines are 0.9985, 0.9999, 0.9994, 0.9997, 0.9995, 0.9995, and 0.999. The respective y-intercepts are $-6.096 \cdot 10^{-5}$, $3.534 \cdot 10^{-5}$, 0.0001049, $-2.306 \cdot 10^{-5}$, $4.244 \cdot 10^{-5}$, 0.0001044, and 0.0001469. The respective R-squared values are 0.9999, 0.9999, 0.9998, 0.9998, 0.9997, 0.9996, and 0.9994. The respective sample sizes are 188,260, 111,897, 13,546, 63,148, 22,640, 29,655, and 10,342. This accounts for about 43.9% of the z_n values between 4,000,001 and 5,000,000. Excluding the groups that almost passed (at n values of the forms p^3qrs and p^2q^2rs) reduces this to about 40.6%. There is little deviation from the reference line for the group at n values of the form p^2qrs. The ratio of the number of z_n values that failed the Lilliefors test to those that passed is about 0.9260 (439,448/474,583). For $4,000,001 \le n \le 5,000,000$, about 91.4% of the z_n values are in the ensemble.

For $5,000,001 \le n \le 6,000,000$, the null hypothesis is rejected by the Lillifors test for z_n values at n values of forms pq, pqr, $pqrs$, p^7q, p^2qrs, p^2q^2rs, and p^2qrst. The respective test statistics and critical values are (0.0034, 0.0020), (0.0024, 0.0019), (0.0036, 0.0026), (0.0315, 0.0314), (0.0046, 0.0035), (0.0089, 0.0087), and (0.0065, 0.0061). The null hypothesis is not rejected by the Shapiro-Wilk test for the group at n values of the form p^7q. See Figures (55) through (61) for quantile-quantile plots of these groups. The respective slopes of the reference lines are 0.9998, 1, 0.9999, 0.9927, 0.9998, 0.9991, and 0.9995. The respective y-intercepts of the reference lines are $-4.225 \cdot 10^{-5}$, $-1.291 \cdot 10^{-5}$, $4.015 \cdot 10^{-5}$, 0.0005051, $6.973 \cdot 10^{-5}$, $9.926 \cdot 10^{-5}$, and 0.0001058. The respective R-squared values are 0.9999, 1, 0.999, 0.9988, 0.9999, 0.9997, and 0.9998. The respective sample sizes are 187,276, 208,604, 113,555, 795, 63,377, 10,400, and 21,108. This accounts for about 60.5% of the z_n values between 5,000,001 and 6,000,000. Excluding the groups that almost passed (at n values of the forms pqr, p^7q, p^2q^2rs, and p^2qrst) reduces this to about 36.4%. There is little deviation from the reference lines for the groups at n values of the forms pq and pqr. The ratio of the number of z_n values that failed the Lilliefors test to those that passed is about 1.9571 (605,115/309,195). For $5,000,001 \leq n \leq 6,000,000$, about 91.4% of the z_n values are in the ensemble.

For $6,000,001 \leq n \leq 7,000,000$, the null hypothesis is rejected by the Lillifors test for z_n values at n values of forms pq, p^5q, $pqrs$, p^2qrs, $p^{11}q$, p^3q^3r, $pqrst$, and $p^3q^2r^2s$. The respective test statistics and critical values are (0.0033, 0.0021), (0.0187, 0.0162), (0.0028, 0.0026), (0.0051, 0.0035), (0.1306, 0.1163), (0.0449, 0.0329), (0.0066, 0.0050), and (0.0403, 0.0394). See Figures (62) through (69) for quantile-quantile plots of these groups. The respective slopes of the reference lines are 0.9999, 0.9975, 0.9999, 0.9997, 0.9393, 0.9909, 0.9995, and 0.9886. The respective y-intercepts of the reference lines are $1.869 \cdot 10^{-5}$, 0.0001429, $2.921 \cdot 10^{-6}$, $5.509 \cdot 10^{-5}$, 0.0004251, -0.0002774, $3.099 \cdot 10^{-5}$, and 0.0007343. The respective R-squared values are 0.9999, 0.9992, 0.9999, 0.9998, 0.9814, 0.9961, 0.9996, and 0.9964. The respective sample sizes are 185,638, 2,996, 114,326, 63,670, 58, 725, 31,253, and 506. This accounts for about 39.9% of the z_n values between 6,000,001 and 7,000,000. Excluding the group that almost passed (at n values of the form $pqrs$) reduces this to about 28.5%. The ratio of the number of z_n values that failed the Lilliefors test to those that passed is about 0.7752 (399,172/514,949). For $6,000,001 \leq n \leq 7,000,000$, about 91.4% of the z_n values are in the ensemble.

For $7,000,001 \leq n \leq 8,000,000$, the null hypothesis is rejected by the Lillifors test for z_n values at n values of forms pq, pqr, $pqrs$, p^2qrs, p^4q^2r, $pqrst$, p^2q^2rs, and p^3q^3rs. The respective test statistics and critical values are (0.0036, 0.0021), (0.0022, 0.0019), (0.0038, 0.0026), (0.0045, 0.0035), (0.0223, 0.0204), (0.0056, 0.0050), (0.114, 0.0088), and (0.0307, 0.0276). See Figures (70) through (77) for quantile-quantile plots of these groups. The respective slopes of the reference lines are 0.9999, 0.9999, 0.9998, 0.9998, 0.9961, 0.9996, 0.9988, and 0.9921. The respective y-intercepts of the reference lines are $2.729 \cdot 10^{-5}$, $-2.123 \cdot 10^{-5}$, $-1.139 \cdot 10^{-5}$, $2.311 \cdot 10^{-5}$, $4.667 \cdot 10^{-5}$, $3.697 \cdot 10^{-5}$, 0.000131, and 0.0006379. The respective R-squared values are 0.9998, 1, 0.9999, 0.9998, 0.9986, 0.9997, 0.999, and 0.9948. The respective sample sizes are 184,760, 208,784, 115,239, 64,000, 1,893, 31,885, 10,232, and 1,034. This accounts for about 61.8% of the

z_n values between 7,000,001 and 8,000,000. Excluding the group that almost passed (at n values of the form pqr) reduces this to about 40.9%. The ratio of the number of z_n values that failed the Lilliefors test to those that passed is about 2.0844 (617,827/296,399). For $7,000,001 \leq n \leq 8,000,000$, about 91.4% of the z_n values are in the ensemble.

For $8,000,001 \leq n \leq 9,000,000$, the null hypothesis is rejected by the Lillifors test for z_n values at n values of forms pq, pqr, $pqrs$, p^2qrs, p^5qr, p^3qrs, p^2qrst, and $p^2q^2r^2s$. The respective test statistics and critical values are (0.0035, 0.0021), (0.0023, 0.0019), (0.0037, 0.0026), (0.0046, 0.0035), (0.0113, 0.0112), (0.0062, 0.0058), (0.0061, 0.0059), and (0.0403, 0.0384). See Figures (78) through (85) for quantile-quantile plots of these groups. The respective slopes of the reference lines are 0.9999, 0.9999, 0.9999, 0.9998, 0.9986, 0.9995, 0.9995, and 0.9879. The respective y-intercepts of the reference lines are $2.113 \cdot 10^{-5}$, $2.059 \cdot 10^{-6}$, $1.198 \cdot 10^{-5}$, $4.654 \cdot 10^{-5}$, 0.0001626, $9.364 \cdot 10^{-5}$, $4.307 \cdot 10^{-5}$, and 0.001257. The respective R-squared values are 0.9999, 1, 0.9999, 0.9999, 0.9997, 0.9996, 0.9997, and 0.9943. The respective sample sizes are 183,836, 308653, 115,969, 64,205, 6,235, 23,148, 22,238, and 531. This accounts for about 62.5% of the z_n values between 8,000,001 and 9,000,000. Excluding the groups that almost passed (at n values of the forms pqr, p^5qr, p^3qrs, and p^2qrst) reduces this to about 36.5%. The ratio of the number of z_n values that failed the Lilliefors test to those that passed is about 2.1599 (624,815/289,277). For $8,000,001 \leq n \leq 9,000,000$, about 91.4% of the z_n values are in the ensemble.

For $9,000,001 \leq n \leq 10,000,000$, the null hypothesis is rejected by the Lillifors test for z_n values at n values of forms pq, p^2q, p^5q, $pqrs$, p^2q^2r, p^3q^2r, p^4q^2r, p^2qrst, and p^4q^2rs. The respective test statistics and critical values are (0.0034, 0.0021), (0.0050, 0.0049), (0.0175, 0.0165), (0.0037, 0.0026), (0.0116, 0.0114), (0.0184, 0.0136), (0.0207, 0.0205), (0.0078, 0.0059), and (0.0196, 0.0167). See Figures (86) through (94) for quantile-quantile plots of these groups. The respective slopes of the reference lines are 0.9998, 0.9996, 0.9972, 0.9998, 0.9986, 0.9976, 0.9962, 0.9995, and 0.9966. The respective y-intercepts of the reference lines are $1.587 \cdot 10^{-5}$, $-1.593 \cdot 10^{-5}$, 0.0002491, $-2.49 \cdot 10^{-5}$, $-5.731 \cdot 10^{-5}$, $-2.883 \cdot 10^{-5}$, $6.592 \cdot 10^{-5}$, $7.014 \cdot 10^{-5}$, and 0.0004226. The respective R-squared values are 0.9998, 0.9999, 0.9987, 0.9999, 0.9996, 0.9985, 0.9989, 0.9996, and 0.9977. The respective sample sizes are 183,425, 32,995, 2,881, 116,952, 6,062, 4,271, 1,860, 22,758, and 2,815. This accounts for about 37.4% of the z_n values between 9,000,001 and 10,000,000. Excluding the groups that almost passed (at n values of the forms p^2q, p^2q^2r, and p^4q^2r) reduces this to about 33.3%. The ratio of the number of z_n values that failed the Lilliefors test to those that passed is about 0.6924 (374,019/540,211). For $9,000,001 \leq n \leq 10,000,000$, about 91.4% of the z_n values are in the ensemble.

Hervé and Molin [4] give the formula $\frac{.895}{f_N}$, $f_N = \frac{.83+N}{\sqrt{N}} - .01$, $N > 50$, where N is the sample size for the critical value of the Lilliefors test at a confidence level of 0.05. For large sample sizes, f_N is approximately equal to $\sqrt{N}$. For $1 \leq n \leq 1,000,000$, $1,000,001 \leq n \leq 2,000,000$, $2,000,001 \leq n \leq 3,000,000$, $3,000,001 \leq n \leq 4,000,000$, $4,000,001 \leq n \leq 5,000,000$, $5,000,001 \leq n \leq 6,000,000$, $6,000,001 \leq n \leq 7,000,000$, $7,000,001 \leq n \leq 8,000,000$, $8,000,001 \leq n \leq 9,000,000$, and $9,000,001 \leq n \leq 10,000,000$ the respective percentages of the z_n values that failed the Lilliefors test (excluding groups that almost passed) are 42.5%, 41.5%, 43.3%, 41.1%, 40.6%, 36.4%, 28.5%, 40.9%, 36.5%, and 33.3%. The rejection rate appears to be decreasing, but the evidence is not that conclusive. The respective percentages of the z_n values that are in the ensemble are 91.0%, 90.4%, 91.4%, 91.4%, 91.4%, 91.4%, 91.4%, 91.4%, 91.4%, and 91.4%. Although this percentage is almost constant, the relative number of z_n values in the group at n values of the form pq (and other groups where n has a simple factorization) is decreasing. See Figure (95) for a plot of the number of z_n values in the groups at n values of the forms pq, p^2q, p^3q, pqr, p^2qr, and p^3qr. Except for the group at n values of the form pqr, the number of z_n values in the groups is decreasing. See Figure (96) for a plot of the number of z_n values in the groups at n values of the forms $pqrs$, p^2qrs, p^3qrs, $pqrst$, p^2q^2rs, and p^2qrst. Except for the group at n values of the form p^2q^2rs, the number of z_n values in the groups is increasing.

4 Ensembles of Normal Probability Distributions with Different Sampling Intervals

For sampling intervals of 1,000,000, the Lilliefors test rejects the null hypothesis for the group of z_n values at n values of the form pq for each interval of n values up to (9,000,001, 10,000,000). For sampling intervals of 1,000,000, the Lilliefors test rejects the null hypothesis for the group of z_n values at n values of the form $pqrs$ for each interval of n values up to (9,000,001, 10,000,000). For sampling intervals of 1,000,000, the Lilliefors test rejects the null hypothesis for the group of z_n values at n values of the form p^2qrs for each interval of n values up to (8,000,001, 9,000,000). Reducing the sampling interval to 250,000 reduces the rejection rate. For the group of z_n values at n values of the form pq and for the corresponding sampling intervals up to (3,750,001, 4,000,000), the test statistics and critical values are (0.0036, 0.0037), (0.0042, 0.0039), (0.0039, 0.0039), (0.0033, 0.0039), (0.0042, 0.0040), (0.0038, 0.0040), (0.0053, 0.0040), (0.0050, 0.0040), (0.0040, 0.0040), (0.0032, 0.0040), (0.0037, 0.0040), (0.0057, 0.0040), (0.0067, 0.0041), (0.0055, 0.0041), (0.0039, 0.0041), and (0.0059, 0.0041). Reducing the sampling interval by 75% reduces the rejection rate to about 43.3% (342,171/790,682). The respective sample sizes are 55,831, 52,368, 51,226, 50,441, 49,817, 49,405, 49,239, 48,733, 48,573, 48,497, 48,082, 47,964, 47,681, 47,816, 47,498, and 47,511. For the group of z_n values at n values of the form $pqrs$ and for the corresponding sampling intervals up to (3,750,001, 4,000,000), the test statistics and critical values are (0.0062, 0.0063), (0.0068, 0.00058), (0.0079, 0.0057), (0.0051, 0.0056), (0.0065, 0.0055), (0.0040, 0.0055), (0.0047, 0.0054), (0.0043, 0.0055), (0.0064, 0.0054), (0.0057, 0.0054), (0.0062, 0.0054), (0.0054, 0.0054), (0.0044, 0.0054), (0.0046, 0.0054), (0.0069, 0.0053), and (0.0041, 0.0053). Reducing the sampling interval by 75% reduces the rejection rate to about 50.3% (209,341/415,950). The respective sample sizes are 20,013, 23,352, 24,460, 25,141, 25,567, 25,901, 26,454, 26,400, 26,763, 27,104, 27,082, 27,304, 27,228, 27,639, 27,709, and 27,833. For the group of z_n values at n values of the form p^2qrs and for the corresponding sampling intervals up to (3,750,001, 4,000,000), the test statistics and critical values are (0.0103, 0.0079), (0.0065, 0.0075), (0.0063, 0.0073), (0.0074, 0.0073), (0.0080, 0.0072), (0.0076, 0.0072), (0.0061, 0.0072), (0.0049, 0.0071), (0.0072, 0.0071), (0.0085, 0.0071), (0.0060, 0.0071), (0.0061, 0.0071), (0.0048, 0.0071), (0.0056, 0.0071), (0.0071, 0.0071), and (0.0062, 0.0071). Reducing the sampling interval by 75% reduces the rejection rate to about 43.1% (103,859/240,963). The respective sample sizes are 12,49, 14,084, 14,535, 148,832, 14,973, 15,142, 15,157, 15,384, 15,386, 15,428, 15,426, 15,579, 15,609, 15,725, 15,619, and 15,605. Reducing the sampling interval to 100,000 reduces the rejection rate more. For the group of z_n values at n values of the form pq and for the corresponding sampling intervals up to (4,900,001, 5,000,000), the rejection rate is about 33.4% (327,316/978,943).

For the group of z_n values at n values of the form p^2q^2rs and for sampling intervals of 1,000,000 up to (4,000,001, 5,000,000), the rejection rate of the Lilliefors test is about 59.8% (31,109/52,064). For the group of z_n values at n values of the form p^2qrst and for sampling intervals of 1,000,000 up to (4,000,001, 5,000,000), the rejection rate is about 35.5% (31,514/88,838). For the group of z_n values at n values of the form pqr and for sampling intervals of 1,000,000 up to (4,000,001, 5,000,000), the rejection rate is about 39.9% (416,036/1,043,468). For the group of z_n values at n values of the form $pqrst$ and for sampling intervals of 1,000,000 up to (4,000,001, 5,000,000), the rejection rate is about 79.0% (101,061/127,887). Reducing the sampling interval to 500,000 reduces the rejection rate. For the group of z_n values at n values of the form p^2q^2rs and for the corresponding sampling intervals up to (4,500,001, 5,000,000), the test statistics and critical values are (0.0135, 0.0125), (0.0103, 0.0122), (0.0102, 0.0122), (0.0066, 0.0122), (0.0121, 0.0122), (0.0074, 0.0122), (0.0137, 0.0123), (0.0076, 0.0122), (0.0103, 0.0123, and (0.0087, 0.0123). The corresponding sample sizes are 5,015, 5,256, 5,255, 5,245, 5,240, 5,166, 5,278, 5,189, and 5,153. Reducing the sampling interval by 50% reduces the rejection rate to about 19.6% (10,181/52,064). For the group of z_n values at n values of the form p^2qrst and for the corresponding sampling intervals up to (4,500,001, 5,000,000), the test statistics and critical values are (0.0092, 0.0122), (0.0121, 0.0103), (0.0086, 0.0097, (0.0060, 0.0094), (0.0078, 0.0092), (0.0075, 0.0091), (0.0065, 0.0090), (0.0062, 0.0088), (0.0054, 0.0088), and (0.0083, 0.0088). The corresponding sample sizes are 5,282, 7,471, 8,288, 8,818, 9,264, 9,497, 9,718, 10,052, 10,201, and 10,247. Reducing the sampling interval by 50% reduces the rejection rate to about 8.4% (7,471/88,838). For the group of z_n values at n values of the form pqr and for the corresponding sampling intervals up to (4,500,001, 5,000,000), the test statistics and critical values are (0.0027, 0.0028), (0.0020, 0.0027), (0.0022, 0.0027), (0.0016, 0.0027), (0.0020, 0.0027), (0.0014, 0.0027), (0.0032, 0.0027), (0.0017, 0.0027), 0.0022, 0.0027), and (0.0019, 0.0027). The corresponding sample sizes are 102,731, 104,233, 104,659, 104,503, 104,354, 104,646, 104,622, 104,450, 104,506, and 104,764. Reducing the sampling interval by 50% reduces the rejection rate to about 10.0% (104,611/1,043,468). For the group of z_n values at n values of the form $pqrst$ and for the corresponding sampling intervals up to (4,500,001, 5,000,000), the test statistics and critical values are (0.0125, 0.0101), (0.0060, 0.0086), (0.0062, 0.0081), (0.0066, 0.0079), (0.0080, 0.0077), (0.0050, 0.0076), (0.0074, 0.0075), (0.0074, 0.0074), (0.0088, 0.0073), and (0.0111, 0.0072). The corresponding sample sizes are 7,713, 10,674, 11,904, 12,664, 13,142, 13,684, 14,115, 14,336, 14,624, and 15,031. Reducing the sampling interval by 50% reduces the rejection rate to about 39.5% (50,510/127,887). Reducing the sampling interval to 200,000 reduces the rejection rate more. For the group of z_n values at n values of the form $pqrst$ and for the corresponding sampling intervals up to (4,800,001, 5,000,000), the rejection rate is about 28.6% (36,577/127,887).

For some groups of z_n values, larger sampling intervals can be used. For n less than or equal to 10,000,000, the null hypothesis is not rejected by the Lillifors test for z_n values at n values of forms p^2q^2, p^3q^2, p^4q^2, $p^2q^2r^2$, and $p^3q^2r^2$. The respective test statistics and critical values are (0.0155, 0.0300), (0.0250, 0.0383), (0.0294, 0.0528), (0.0271, 0.0393), and (0.0369, 0.0382). The respective sample sizes are 875, 536, 282, 509, and 539.

5 Another Ensemble of Normal Probability Distributions

The ensemble consists of 83 groups of z_n values at n values of the forms p^3, p^4, p^3q^3, p^5q^2, p^6q^2, p^5q^3, p^7q^2, $p^{12}q$, p^6q^3, p^4q^4, $p^{13}q$, $p^{14}q$, p^9q^2, p^8q^2, p^5q^4, p^7q^2, $p^{10}q^2$, p^6q^4, $p^{16}q$, $p^{15}q$, $p^3q^3r^2$, $p^4q^2r^2$, $p^{10}qr$, $p^{11}q^2$, p^8q^3, p^7q^4, p^6q^5, p^5q^5, $p^{17}q$, $p^{18}q$, $p^{11}qr$, $p^{12}q^2$, p^9q^3, p^8q^4, p^7q^5, $p^4q^3r^2$, $p^5q^2r^2$, $p^{12}qr$, $p^{13}q^2$, $p^{10}q^3$, p^9q^4, p^8q^5, p^8q^2r, $p^3q^3r^3$, $p^{19}q$, p^7q^3r, $p^6q^2r^2$, p^9q^2r, p^7q^6, $p^{10}q^4$, $p^{11}q^3$, $p^{14}q^2$, $p^{13}qr$, p^8qrs, $p^{10}q^2r$, p^8q^3r, $p^5q^3r^2$, p^7q^2r, $p^{14}qr$, $p^{12}q^3$, and $p^{15}q^2$, $p^{19}q$, p^9qrs, p^3q^2rst, $p^{11}q^2r$, p^9q^3r, $p^8q^2r^2$, $p^7q^3r^2$, p^6q^2rs, $p^4q^2r^2s$, $p^2q^2r^2r^2$, $p^{14}qr$, p^4qrst, $p^2q^2r^2st$, $p^5q^2q^2s$, p^5q^2rs, p^4q^3rs, p^4q^4rs, $p^4q^2r^2s$, p^7q^2rs, and p^2qrstu. For $1 \leq n \leq 10,000,000$, the respective sample sizes are 47, 16, 65, 178, 117, 39, 82, 367, 28, 15, 200, 111, 41, 57, 20, 20, 29, 14, 35, 61, 184, 285, 1,923, 21, 15, 10, 8, 6, 20, 11, 972, 15, 11, 8, 6, 188, 164, 477, 11, 9, 5, 4, 1,351, 19, 7, 733, 100, 717, 4, 4, 6, 8, 239, 3,911, 382, 380, 108, 62, 116, 6, 4, 4, 6, 7, 1,775, 25830, 203, 199, 39, 43, 5,583, 2,200, 93, 116, 24,192, 4,052, 988, 3,226, 7,791, 1,355, 2,200, 2,586, and 22,852. For $1 \leq n \leq 10,000,000$, the null hypothesis is rejected by the Lillifors test for z_n values at n values of the forms $p^{12}q$, $p^{13}q$, $p^{14}q$, $p^3q^3r^2$, $p^4q^2r^2$, p^8q^4, p^7q^5, $p^4q^3r^2$, $p^5q^2r^2$, p^9q^4, p^8q^5, $p^6q^2r^2$, $p^{10}q^4$, $p^{11}q^3$, $p^5q^3r^2$, $p^7q^2r^2$, $p^{12}q^3$, $p^{11}q^4$, p^9q^5, p^3q^2rst, $p^7q^3r^2$, p^4qrst, $p^2q^2r^2st$, and p^2qrstu. The respective test statistics and critical values are (0.0505, 0.0462), (0.0666, 0.0626), (0.0854, 0.0841), (0.0896, 0.0653), (0.0540, 0.0525), (0.2909, 0.2850), (0.3477, 0.3190), (0.1011, 0.0646), (0.0730, 0.0692), (0.3960, 0.3370), (0.04100, 0.3810), (0.1012, 0.0886), (0.4100, 0.3810), (0.3240, 0.3190), (0.1225, 0.0853), (0.1251, 0.1125), (0.3240, 0.3190), (0.4100, 0.3810), (0.4100, 0.3810), (0.0070, 0.0055), (0.1725, 0.1351), (0.0063, 0.0057), (0.0157, 0.0139), and (0.0086, 0.0059). See Figures (97) through (120) for the quantile-quantile plots of these sorted z_n values and the corresponding F^{-1} values. The respective slopes of the reference lines are 0.9839, 0.9725, 0.9538, 0.9679, 0.9796, 0.7616, 0.6847, 0.9741, 0.9729, 0.6219, 0.5832, 0.96, 0.5832, 0.7091, 0.9583, 0.9413, 0.7091, 0.5832, 0.5832, 0.9996, 0.9142, 0.9995, 0.9979, 0.9995. The respective y-intercepts of the reference lines are 0.002664, 0.004333, 0.008823, 0.003351, 0.001379, 0.0005612, -0.009821, 0.002242, 0.003757, 0.02662, 0.06879, 0.005686, 0.06879, 0.04205, 0.00696, 0.009502, 0.04205, 0.06879, 0.06879, $7.249 \cdot 10^{-5}$, 0.01255, $5.581 \cdot 10^{-5}$, -0.0001115, and $-8.515 \cdot 10^{-6}$. The respective R-squared values are 0.9926, 0.9854, 0.9707, 0.9788, 0.9898, 0.8905, 0.7853, 0.9906, 0.9931, 0.6898, 0.6603, 0.9887, 0.6603, 0.8425, 0.9811, 0.9805, 0.8031, 0.6603, 0.6603, 0.9998, 0.9545, 0.9998, 0.9991, and 0.9998. The respective sample sizes are 367, 200, 111, 184, 285, 8, 6, 188, 164, 5, 4, 100, 4, 6, 108, 62, 6, 4, 4, 25,830, 43, 24,192, 4,052, and 22,852. Some of the groups with small sample sizes have the same z_n values. Excluding the groups at n values of the forms p^3q^2rst, p^4qrst, and p^2qrstu, the rejection rate is about 12.71% (5,911/46,518). The groups at n values of the forms p^3q^2rst, p^4qrst, and p^2qrstu have a combined sample size of 72,874.

6 Minima and Maxima in the $\sum_{i|n}(\gamma_{i+1} - \gamma_i)\mu(i)$ Distribution

Let l_n denote the minimum of the $\sum_{i|n}(\gamma_{i+1} - \gamma_i)\mu(i)$ distribution. l_n is a slowly decreasing step function with an initial value of $\gamma_2 - \gamma_1$. For n less than 500,000, l_n decreases in value at $n = 2$, 3, 6, 15, 195, 435, 615, 1590, 4305, 4,935, 7,995, 17,355, 32,595, 72,615, 228,165, and 261,555. The prime factorizations of these values are 2, 3, $2 \cdot 3$, $3 \cdot 5$, $3 \cdot 5 \cdot 13$, $3 \cdot 5 \cdot 29$, $3 \cdot 5 \cdot 41$, $2 \cdot 3 \cdot 5 \cdot 53$, $3 \cdot 5 \cdot 7 \cdot 41$, $3 \cdot 5 \cdot 7 \cdot 47$, $3 \cdot 5 \cdot 13 \cdot 41$, $3 \cdot 5 \cdot 13 \cdot 89$, $3 \cdot 5 \cdot 41 \cdot 53$, $3 \cdot 5 \cdot 47 \cdot 103$, $3 \cdot 5 \cdot 7 \cdot 41 \cdot 53$, and $3 \cdot 5 \cdot 7 \cdot 47 \cdot 53$ respectively. The values are square-free and the number of prime factors is non-decreasing. The corresponding l_n values are 2.898497, 1.473297, 0.817020, -1.210552, -2.2385884, -2.584503, -3.612455, -3.717184, -3.834182, -4.450978, -5.171634, -5.675754, -6.993299, -7.754984, -8.439405, and -8.711062. The values of n that are the products of 2, $2 \cdot 3$, $2 \cdot 3 \cdot 5$, $2 \cdot 3 \cdot 5 \cdot 7$, $2 \cdot 3 \cdot 5 \cdot 7 \cdot 11$, ... are 2, 6, 30, 210, 2,310, 30,030, ... The above value of 195 is between 30 and 210, the above values of 435, 615, 1,590 are between 210 and 2,310, the above values of 4,305, 4,935, 7995, and 17,355 are between 2,310, and 30,030, etc. This is useful in determining how frequently l_n will decrease in value.

Let g_n denote the maximum of the $\sum_{i|n}(\gamma_{i+1} - \gamma_i)\mu(i)$ distribution. g_n is a slowly increasing step function with an initial value of $\gamma_2 - \gamma_1$. For n less than 500,000, g_n increases in value at n = 2, 1,806, 6,118, 17,822, 25,802, 34,314, 70,518, 131,838, 186,018, 204,078, 213,486, and 364,182. The prime factorizations of these values are 2, $2 \cdot 3 \cdot 7 \cdot 43$, $2 \cdot 7 \cdot 19 \cdot 23$, $2 \cdot 7 \cdot 19 \cdot 67$, $2 \cdot 7 \cdot 19 \cdot 97$, $2 \cdot 3 \cdot 7 \cdot 19 \cdot 43$, $2 \cdot 3 \cdot 7 \cdot 23 \cdot 73$, $2 \cdot 3 \cdot 7 \cdot 43 \cdot 73$, $2 \cdot 3 \cdot 7 \cdot 43 \cdot 107$, $2 \cdot 3 \cdot 7 \cdot 13 \cdot 17 \cdot 23$, and $2 \cdot 3 \cdot 7 \cdot 13 \cdot 23 \cdot 29$. The values are square-free and the number of prime factors is non-decreasing. The corresponding g_n values are 6.887315, 6.938951, 7.786355, 7.926703, 8.147173, 8.684880, 8.930729, 8.999220, 9.061730, 9.071198, 10.498812, and 10.520155.

7 The Distribution of the $\sum_{i|n} \gamma_i \mu(i)$ Values

Let y_n denote $\sum_{i|n} \gamma_i \mu(i)$. See Figure (121) for a plot of the distribution of y_n values for $n \leq 1,000$. For $n \leq 100,000$, the mean of the y_n distribution is -30.4965 and the standard deviation is 25,342. See Figure (122) for a histogram (with 240 bins) of these values overlain with a histogram of randomly generated values for a normal probability distribution with the same mean and standard deviation. (The values have been scaled by a factor of 0.001 prior to histogramming and offset by 120 bins to the right.) The histogram of the y_n values has a large spike in the center relative to the histogram of the normal probability distribution, so the y_n values are poorly modeled by a normal probability distribution. For $n \leq 999$, the mean of the y_n distribution is -8.4643 and the standard deviation is 473.2557. See Figure (123) for a plot of the sorted y_n values and $F^{-1}(p) = \mu + \sigma \Phi^{-1}(p)$ (where $\mu = -8.4643$ and $\sigma = 473.2557$) for p = 0.001, 0.002, 0.003, ..., 0.999. The minimum and maximum y_n values are $-1,401.6$ and 1,206.9 and the minimum and maximum F^{-1} values are $-1,470.9$ and 1,454.0. For larger samples of zeros, the y_n values are similarly bounded by the generalized probit function. See Figure (124) for a plot of the means of the y_n distributions for n less than or equal to 1,000, 2,000, 3,000, ..., and 10,000,000. There doesn't appear to be any way to predict what the means will be, but they appear to increase more slowly than the Mertens function (the summatory Möbius function). Let $M(n)$ denote the Mertens function. See Figure (125) for a plot of $M(n)$, n = 1, 2, 3, ..., 1,000, and a plot of the means of the y_n distributions for n less than or equal to 1, 2, 3, ..., and 1,000. The peaks and valleys of the two curves occur at roughly the same places. See Figure (126) for a plot of $M(n)$, n = 1, 2, 3, ..., 2,000,000, and a plot of the means of the y_n distributions for n less than or equal to 1, 2, 3, ..., and 2,000,000. See Figure (127) for a plot of the absolute values of the means and $\sqrt{n}$ for $n \leq 100,000$. See Figure (128) for a plot of the standard deviations of the y_n distributions for n less than or equal to 1,000, 2,000, 3,000, ..., and 10,000,000. For a cubic least-squares fit of these values, $p_1 = 1.704 \cdot 10^{-16}$ with a 95% confidence interval of $(1.687 \cdot 10^{-16}, 1.72 \cdot 10^{-16})$, $p_2 = -4.109 \cdot 10^{-9}$ with a 95% confidence interval of $(-4.134 \cdot 10^{-9}, -4.084 \cdot 10^{-9})$, $p_3 = 0.1923$ with a 95% confidence interval of (0.1922, 0.1924), $p_4 = 1.262 \cdot 10^4$ with a 95% confidence interval of $(1.249 \cdot 10^4, 1.274 \cdot 10^4)$, SSE=$2.542 \cdot 10^{10}$, R-squared=1, and RMSE=1,595. The curve of standard deviations is almost quadratic. See Figure (129) for a plot of the maximum y_n values for n less than or equal to 1,000, 2,000, 3,000, ..., and 10,000,000. For a cubic least-squares fit of these values, SSE=$2.139 \cdot 10^{11}$, R-squared=1, and RMSE=4,625. See Figure (130) for the minimum y_x values for n less than or equal to 1,000, 2,000, 3,000, ..., and 10,000,000. For a cubic least-squares fit of these values, SSE=$2.234 \cdot 10^{11}$, R-squared=1, and RMSE=4,728.

See Figure (131) for a linear least-squares fit of y_n for prime n less than $10,000,000$. The slope is -7.5, the y-intercept is 13,440, R-squared=1, and the sample size is 664,579. See Figure (132) for a linear least-squares fit of the y_n values at n values of the form p^2 for $n \leq 10,000,000$. The slope is -8.146, the y-intercept is -22.09, R-squared=0.9999, and the sample size is 446. See Figure (133) for a linear least-squares fit of the y_n values at n values of the form p^3 for $n \leq 10,000,000$. The slope is -8.597, the y-intercept is 4.08, R-squared=0.999, and the sample size is 47. See Figure (134) for a linear least-squares fit of the y_n values at n values of the form p^4 for $n \leq 10,000,000$. The slope is -8.795, the y-intercept is 6.205, R-squared=0.9974, and the sample size is 16. In general, the y_n values at prime powers of n appear to be almost linear. All the y_n values can then be grouped into approximately linear curves. For example, see the plot of the 875 y_n values at n values of the form p^2q^2 for $n \leq 10,000,000$ (Figure 135). See Figures (136) through (141) for the respective linear least-squares fits of the y_n values at n values of the forms 2^2q^2, 3^2q^2 $(q > 3)$, 5^2q^2 $(q > 5)$, 7^2q^2 $(q > 7)$, 11^2q^2 $(q > 11)$, and 13^2q^2 $(q > 13)$. The respective slopes are 6.57, 12.65, 23.97, 34.51, 53.7, and 63.32. See Figure (142) for a quadratic least-squares fit of these values. The respective y-intercepts are -12.9, -10.63, 3.728, 41.19, 134.3, and 214.6. See Figure (143) for a quadratic least-squares fit of these values. For another example, see the plot of the 47,601 y_n values at n values of the form p^3q^2r for $n \leq 10,000,000$ (Figure 144). See Figure (145) for a plot of the 12,935 y_n values where $3^2 \cdot 2$ divides n and 3^3 does not divide n. Twenty of these n values are of the form 5^33^22, 7^33^22, 11^33^22, ..., 79^33^22. See Figure (146) for a linear least-squares fit of the y_n values at the remaining 12,915 n values. The slope is -11.71, the y-intercept is 957.4, and R-squared=1. See Figure (147) for a linear least-squares fit of the y_n values at the 20 n values. The slope is -9.278, the y-intercept is -1.726, and R-squared=0.9978. See Figure (148) for a plot of the 8,962 y_n values where $2^2 \cdot 3$ divides n and 2^3 does not divide n. Twenty-two of these n values are of the form 5^32^23, 7^32^23, 11^32^23, ..., 83^32^23. See Figure (149) for a linear least-squares fit of the y_n values at the remaining 8,940 n values. The slope is -11.64, the y-intercept is 717.4, and R-squared=1. See Figure (150) for a linear least-squares fit of the y_n values at the 22 n values. The slope is -9.273, the y-intercept is -1.755, and R-squared=0.9983. See Figure (151) for a plot of the 5,145 y_n values where $5^2 \cdot 2$ divides n and 5^3 does not divide n. Thirteen of these n values are of the form 7^35^22, 11^35^22, ..., 53^35^22. See Figure (152) for a linear least-squares fit of the y_n values at the remaining 5,132 n values. The slope is -22.39, the y-intercept is 1016, and R-squared=0.9999. See Figure (153) for a linear least-squares fit of the y_n values at the 13 n values. The slope is -17.39, the y-intercept is -20.64, and R-squared=0.9962.

8 The Convolution of the Differences of the Eigenvalues of a Gaussian-Random Hermitian Matrix with the Möbius Function

Let e_1, e_2, e_3, ..., e_d denote the eigenvalues of a dxd Gaussian-random Hermitian matrix. Such matrices where $d = 1,000$, 2,000, 3,000, and 4,000 were generated using R. (The diagonals were scaled by a factor of $\sqrt{2}$.) Let w_n denote $\sum_{i|n}(e_{i+1} - e_i)\mu(i)$. For a normal probability fit of the 288 w_n values for $n \leq 999$ (for $d = 1,000$) and at n values of the form pq, the mean is 1.2332 with a 95% confidence interval of (1.2119, 1.2544) and the standard deviation is 0.1831 with a 95% confidence interval of (0.1692, 0.1994). See Figure (154) for a quantile-quantile plot of the sorted w_n values and the corresponding F^{-1} values. For a linear least-squares fit of the curve, the slope is 0.98 with a 95% confidence interval of (0.9686, 0.9915), the y-intercept is 0.0264 with a 95% confidence interval of (0.01032, 0.03896), and R-squared 0.99. The null hypothesis is not rejected by the Lilliefors test. The test statistic and critical value are (0.0328, 0.0522). For w_n values at n values of the forms p^2, p^2q, p^3q, pqr, p^4q, p^2q^2, p^5q, p^3q^2, p^2qr, p^6q, p^4q^2, p^3qr, $pqrs$, p^2q^2r, p^4qr, p^3q^2r, and p^2qrs, the respective test statistics and critical values are (0.1509, 0.2490), (0.0673, 0.0853), (0.0974, 0.1336), (0.0579, 0.0763), (0.1097, 0.1866), (0.1658, 0.3000), (0.1603, 0.2490), (0.1731, 0.3000), (0.0761, 0.0997), (0.1713, 0.3370), (0.2911, 0.3810), (0.1128, 0.1682), (0.1234, 0.2130), (0.3154, 0.2340), (0.2042, 0.2710), (0.3985, 0.3000), and (0.3015, 0.3190). The null hypothesis is rejected by the Lilliefors test for groups of w_n values of the forms p^2q^2r and p^3q^2r. See Figures (155) through (171) for the corresponding quantile-quantile plots. The respective slopes of the linear least-squares fits are 0.8277, 0.962, 0.9308, 0.969, 0.8876, 0.783, 0.8279, 0.7684, 0.9518, 0.7349, 0.6764, 0.898, 0.8567, 0.7762, 0.8056, 0.6516, and 0.7048. The respective y-intercepts of the linear least-square fits are 0.2223, 0.04674, 0.08832, 0.03997, 0.1504, 0.2799, 0.2448, 0.3289, 0.06899, 0.3671, 0.467, 0.1488, 0.2184, 0.3285, 0.2978, 0.5113, and 0.4707. The respective R-squared values are 0.9694, 0.9888, 0.9875, 0.9927, 0.9841, 0.9785, 0.9701, 0.9428, 0.9853, 0.963, 0.888, 0.9714, 0.9609, 0.8217, 0.9653, 0.6774, and 0.8322. The respective sample sizes are 11, 108, 44, 135, 21, 7, 11, 7, 79, 5, 4, 27, 16, 13, 9, 7, and 6.

For a normal probability fit of the 563 w_n values for $n \leq 1,999$ (for $d = 2,000$) and at n values of the form pq, the mean is -0.0393 with a 95% confidence interval of (-0.0503, -0.0283) and the standard deviation is 0.1330 with a 95% confidence interval of (0.1256, 0.1412). See Figure (172) for a quantile-quantile plot of the sorted w_n values and the corresponding F^{-1} values. For a linear least-squares fit of the curve, the slope is 0.982 with a 95% confidence interval of (0.9706, 0.9933), the y-intercept is -0.0007254 with a 95% confidence interval of (-0.002298, 0.0008467), and R-squared 0.9809. The null hypothesis is rejected by the Lilliefors test. The test statistic and critical value are (0.0468, 0.0373). For w_n values at n values of the forms p^2, p^2q, p^3q, pqr, p^4q, p^2q^2, p^5q, p^3q^2, p^2qr, p^6q, p^4q^2, p^3qr, $pqrs$, p^7q, p^2q^2r, p^4qr,

$p3q^2r$, p^2qrs, p^5qr, and p^4q^2r, the respective test statistics and critical values are (0.1163, 0.2270), (0.0886, 0.0639), (0.1052, 0.0991), (0.0424, 0.0510), (0.1082, 0.1419), (0.1257, 0.2420), (0.1593, 0.1900), (0.1872, 0.2490), (0.0565, 0.0670), (0.1446, 0.2490), (0.3138, 0.3370), (0.0906, 0.1144), (0.0797, 0.1306), (0.1859, 0.3370), (0.0999, 0.1706), (0.1406, 0.1798), (0.2517, 0.2000), (0.2105, 0.1682), (0.1876, 0.2850), and (0.3844, 0.3190). The null hypothesis is rejected for groups of w_n values at n values of the forms p^2q, p^3r, p^3q^2r, p^2qrs, and p^4q^2r. See Figures (173) through (192) for the corresponding quantile-quantile plots. The respective slopes of the linear least-squares fits are 0.8489, 0.9625, 0.9476, 0.9818, 0.9217, 0.8342, 0.8736, 0.8146, 0.9689, 0.8313, 0.7023, 0.939, 0.9303, 0.7355, 0.898, 0.8873, 0.8444, 0.8675, 0.7938, and 0.6765. The respective y-intercepts of the linear least-square fits are 0.01091, −0.003082, −0.004669, −0.004108, −0.00605, −0.02941, −0.01322, −0.02984, −0.007276, −0.01911, −0.06441, −0.01493, −0.02285, −0.04703, −0.02916, −0.02969, −0.05067, −0.04678, −0.07093, and −0.1071. The respective R-squared values are 0.9682, 0.9666, 0.9759, 0.9929, 0.98, 0.9658, 0.9608, 0.9397, 0.9829, 0.9781, 0.8817, 0.9779, 0.9818, 0.9649, 0.9761, 0.9696, 0.9134, 0.9066, 0.9675, and 0.7667. The respective sample sizes are 14, 192, 80, 302, 39, 12, 20, 11, 175, 11, 5, 60, 46, 5, 26, 23, 18, 27, 8, and 6.

Satisfactory sampling intervals are much smaller than for the zeta function zeros. For the above groups of w_n values and for $d = 3,000$, the corresponding test statistics and critical values are (0.0938, 0.0308), (0.1553, 0.2130), (0.1027, 0.0536), (0.1381, 0.0833), (0.0787, 0.0403), (0.926, 0.1206), (0.2054, 0.2270), (0.1215, 0.1682), (0.3266, 0.2340), (0.0924, 0.0541), (0.1974, 0.2270), (0.2808, 0.2850), (0.1066, 0.0914), (0.0646, 0.0973), (0.2658, 0.2850), (0.1151, 0.1336), (0.1386, 0.1437), (0.2138, 0.1634), (0.1103, 0.1266), (0.1351, 0.2130), and (0.2508, 0.2490). The respective sample sizes are 825, 16, 273, 113, 483, 54, 14, 27, 13, 268, 14, 8, 94, 83, 8, 44, 38, 29, 49, 16, and 11. For $d = 4,000$, the corresponding test statistics and critical values are (0.1046, 0.0269), (0.1586, 0.2000), (0.1317, 0.0474), (0.0843, 0.0741), (0.0526, 0.0343), (0.0592, 0.1074), (0.1953, 0.2000), (0.1038, 0.1519), (0.1900, 0.2200), (0.0858, 0.0466), (0.1471, 0.1950), (0.1990, 0.2710), (0.0777, 0.0774), (0.0525, 0.0796), (0.2481, 0.2580), (0.0901, 0.1144), (0.0971, 0.1206), (0.1197, 0.1437), (0.0850, 0.1016), (0.1229, 0.1832), and (0.1996, 0.2270). The respective sample sizes are 1,087, 18, 349, 143, 667, 68, 18, 34, 15, 361, 19, 9, 131, 124, 10, 60, 54, 38, 76, 22, and 14. The sample interval becomes too large for the groups of w_n values at n values of the forms pq, p^2q, and p^3q after $d = 1,000$. The sample interval becomes too large for the groups of w_n values at n values of the forms pqr, p^2qr, and p^3qr after $d = 2,000$.

See Figure (193) for a plot of $\sum_{i|n} e_i\mu(i)$ for n = 1, 2, 3, ..., 4,000. The curves are not as linear as for y_n. See Figures (194) and (195) for the means and standard deviations of the $\sum_{i|n} e_i\mu(i)$ distributions for $n \leq 4,000$. Unlike the y_n distributions, the means become approximately equal to zero.

9 Ensembles of Non-Trivial Zeta Function Zeros

See Figure (196) for a linear least-squares fit of the 664,579 γ_n values at prime n, $n \leq 10,000,000$. The slope is 7.5 with a 95% confidence interval of (7.5, 7.5), the y-intercept is 13,450 with a 95% confidence interval of (13,450, 13,450), SSE=8.586·10^{12}, R-squared=1, and RMSE=3,594. See Figure (197) for a linear least-squares fit of the 348,512 γ_n values at n values of the form $2p$, $p > 2$, $n \leq 10,000,000$. The slope is 14.32 with a 95% confidence interval of (14.32, 14.32), the y-intercept is 6,821 with a 95% confidence interval of (6,808, 6,833), SSE=1.281·10^{12}, R-squared=1, and RMSE=1,917. Similar linear least-squares fits are obtained at n values of the forms $3p$, $(p > 3)$, $5p$, $(p > 5)$, $7p$, $(p > 7)$, ... so that linear least-squares fits are obtained at n values of the form pq. Similar result are obtained for other factors.

See Figure (198) for a quadratic least-squares fit of the 446 γ_n values at n values of the form p^2, $n \leq 10,000,000$. SSE=1.775·10^{11}, R-squared=0.9998, and RMSE=20,020. See Figure (199) for a quadratic least-squares fit of the 248 γ_n values at n values of the form $4p^2$, $p > 2$, $n \leq 10,000,000$. SSE=2.361·10^{11}, R-squared=0.9996, and RMSE=31,050. Similar results are obtained for other factors where the exponents of the prime divisors of the factor are greater than or equal to 2.

See Figure (200) for a cubic least-squares fit of the 47 γ_n values at n values of the form p^3, $n \leq 10,000,000$. SSE=1.986·10^{11}, R-squared=0.9976, and RMSE=67,950. See Figure (201) for a cubic least-squares fit of the 27 γ_n values at n values of the form $8p^3$, $p > 2$, $n \leq 10,000,000$. SSE=2.507·10^{11}, R-squared=0.9959, and RMSE=104,400. Similar results are obtained for other factors where the exponents of the prime divisors of the factor are greater than or equal to 3.

The Dirichlet products of the non-trivial zeta function zeros can be similarly grouped.

10 Dirichlet Products of Non-Trivial Zeta Function Zeros and Möbius Inversion

Let $\kappa_1(1)$, $\kappa_1(2)$, $\kappa_1(3)$, ..., $\kappa_1(n)$ denote γ_1, γ_2, γ_3, ..., γ_n and let $\kappa_m(n)$, $m = 2, 3, 4, ...$, denote these values and $m - 1$ values that have been linearly interpolated between successive values. (The powers of the primes can be derived using a Fourier transform of the Riemann spectrum. [See chapter 35 of Mazur and Stein's [5] book.] For 1,000 terms of the spectrum, the transform is $-\sum_1^{1000} \cos(\log(s)\gamma_i)$. The powers of the primes can also be derived using the "interpolated spectrum". The transform is $-\sum_1^C \cos(\log(m \cdot s)\kappa(m)_i)$ where C is the cutoff point. The reciprocal of the s increment must be exactly divisible by m. See Figure (251) for a plot of $-\sum_1^{100000} \cos(\log(s)\gamma_i)$ for $s = 0.1$, 0.2, 0.3, ... 30.0. Spikes occur at the positions of 20, 30, 40, 50, 70, 80, 90, 110, 130, 160, 170, 190, 230, 250, 270, and 290. See Figure (252) for a plot of $-\sum_1^{100000} \cos(\log(2 \cdot s)\kappa(2)_i)$. Spikes occur at the positions of 10, 15, 20, 25, 35, 40, ..., and 295. See Figure (253) for a plot of $-\sum_1^{100000} \cos(\log(5 \cdot s)\kappa(5)_i)$. Spikes occur at the positions of 4, 6, 8, 10, 14, 16, ..., and 298. See Figure (254) for a plot of $\pi(x)$ and $R(x)$ where 2 values have been linearly interpolated between each pair of successive zeros. Out of the 600 "zeros" used, only 200 of them are actual zeros.) See Figure (202) for a plot of $\sum_{i|n} \kappa_1(n/i)$ and $\sum_{i|n} \kappa_5(n/i)$ at n values of the form $n = p^3q^2r$ for $n \le 20,000$. Note that only 4,000 γ values are used to compute $\sum_{i|n} \kappa_5(n/i)$. See Figure (203) for plots of the normalized and scaled values. The curves are normalized by subtracting the first element of each curve. The $\sum_{i|n} \kappa_5(n/i)$ values are then scaled by multiplying them by the ratio of the last elements of the normalized curves. The resulting curves are almost the same. See Figure (204) for a plot of the relative errors of the scaled $\sum_{i|n} \kappa_5(n/i)$ values. For this method to be of value, the scaling factor would have to be somehow predicted.

See Figure (205) for a plot of the 11 $\sum_{i|n} \kappa_1(n/i)$ values at prime n for $n \le 32$. For a linear least-squares fit of the values, the slope is 8.64 with a 95% confidence interval of (8.148, 9.131), the intercept is 22.73 with a 95% confidence interval of (19.39, 26.06), SSE=46.76, R-squared=0.9943, and RMSE=2.279. There are 18 primes less than 64, so the estimated $\sum_{i|n} \kappa_1(n/i)$ value at the largest of these primes is 178.25 (8.64·18+22.73). The actual value is 179.671794, so the ratio of the values is about 0.992087. For a linear least-squares fit of the 18 $\sum_{i|n} \kappa_1(n/i)$ values at prime n for $n \le 64$, the slope is 8.84 and the intercept is 21.8. There are 31 primes less than 128, so the estimated $\sum_{i|n} \kappa_1(n/i)$ value at the largest of these primes is 295.84 (8.84·31+21.8). The actual value is 296.59984, so the ratio of the values is about 0.99744. See Figure (206) for a plot of these and similar ratios computed for n less than or equal to 128, 256, 512, ..., 8,388,608. After the first few values, the ratios approach 1 from above. This is then useful for computing scaling factors. See Figure (207) for a plot of the slopes of these linear least-squares fits. See Figure (208) for a plot of the intercepts of the linear least-squares fits. The slopes slowly become smaller and the intercepts rapidly become larger. See Figure (209) for a plot of the slopes versus $1/\log(\pi(n))$ for $n = 5,000$, 10,000, 15,000, ... 6,000,000 (where $\pi(n)$ is the number of primes less than or equal to n). For a quadratic least-squares fit of the values, $p_1 = -30.28$ with a 95% confidence interval of $(-31.31, -29.25)$, $p_2 = 14.04$ with a 95% confidence interval of (13.85, 14.22), $p_3 = 6.622$ with a 95% confidence interval of (6.614, 6.631), SSE=$2.541 \cdot 10^{-5}$, R-squared=0.9999, and RMSE=0.0004661. See Figure (210) for a plot of the intercepts versus $\sqrt{\pi}(n)$. For a quadratic least-squares fit of the values, $p_1 = 0.01679$ with a 95% confidence interval of (0.01666, 0.01692), $p_2 = 3.177$ with a 95% confidence interval of (3.071, 3.283), $p_3 = -129.3$ with a 95% confidence interval of $(-149.3, -109.4)$, SSE=$2.843 \cdot 10^4$, R-squared=1, and RMSE=15.59.

For the $\sum_{i|n} \kappa_2(n/i)$ values for $n \leq 64$, the scaling factor is 1.66103279905 ((178.25-35.15675)/(117.860263-31.713108)). 178.25 is the estimated value for $\sum_{i|n} \kappa_1(n/i)$ at the largest prime n value less than 64, 35.15675 is the value of $\sum_{i|n} \kappa_1(n/i)$ at $n = 2$, 117.860263 is the value of $\sum_{i|n} \kappa_2(n/i)$ at the largest prime n value less than 64, and 31.713108 is the value of $\sum_{i|n} \kappa_2(n/i)$ at $n = 2$. The $\sum_{i|n} \kappa_2(n/i)$ values are normalized (by subtracting 31.713108), multiplied by the scaling factor, and then re-normalized (by adding 31.713108). See Figure (211) for a plot of the resulting $\sum_{i|n} \kappa_2(n/i)$ values and the $\sum_{i|n} \kappa_1(n/i)$ values. See Figure (212) for a plot of the relative errors. The relative errors are greater than -20% and usually less than 10%. The $\sum_{i|n} \kappa_2(n/i)$ value at $n = 1$ is set to γ_1 before Möbius inversion. See Figure (213) for a plot of the Möbius inverse of the $\sum_{i|n} \kappa_2(n/i)$ values and the actual γ values. See Figure (214) for a plot of the relative errors of the γ values (for $n > 32$). The relative errors range from about -3.44% to 18.63%. See Figures (215) through (228) for plots of the corresponding relative errors for n equal to 128, 256, 512, ..., and 1,048,576. Respective ranges of the relative errors are (-1.48%, 13.37%), (-0.44%, 9.95%), (2.12%, 8.73%), (1.15%, 5.58%), (1.27%, 4.25%), (1.26%, 3.27%), (0.80%, 2.27%), (1.08%, 2.19%), (0.77%, 1.63%), (0.74%, 1.44%), (0.71%, 1.29%), (0.65%, 1.14%), (0.66%, 1.09%), and (0.58%, 0.95%). The ranges of the relative errors for n equal to 2,097,152, 4,194,304, and 8,388,608 are (0.55%, 0.89%), (0.52%, 0.82%), and (0.48%, 0.76%) respectively.

11 Dirichlet Products of Sub-Curves of Non-Trivial Zeta Function Zeros

For a linear least-squares fit of the 53 $\sum_{i|n} \kappa_1(n/i)$ values at $n = 2p$ ($p \neq 2$) and for $n \leq 512$, the slope is 23.59 with a 95% confidence interval of (23.4, 23.79), the intercept is 73.16 with a 95% confidence interval of (67.17, 79.14), SSE=5,840, R-squared=0.9992, and RMSE=10.7. For $n \leq 1,024$, there are 96 $\sum_{i|n} \kappa_1(n/i)$ values at $n = 2p$, so the estimated value of $\sum_{i|n} \kappa_1(n/i)$ at the largest $n = 2p$ value is 2337.8 (23.59·96+73.16). The actual value is 2296.977315. The scaling factor for $\sum_{i|n} \kappa_2(n/i)$ values at $n = pq$ and for $n \leq 1,024$ is then 1.78817824 ((2337.8-97.753801)/(1333.150397-80.453014)). 97.753801 is the $\sum_{i|n} \kappa_1(n/i)$ value at $n = 6$, 1333.150397 is the $\sum_{i|n} \kappa_2(n/i)$ value at the largest $n = 2p$ value for $n \leq 1,024$, and 80.453014 is the $\sum_{i|n} \kappa_2(n/i)$ value at $n = 6$. Möbius inversion is then done using this scaling factor and normalization value for all $\sum_{i|n} \kappa_2(n/i)$ values for $n \leq 1,024$ and the estimated γ values at $n = pq$ are extracted. See Figure (229) for a plot of the relative errors for the last 144 ($293 - 149$) estimated values. (There are 293 γ values at $n = pq$ for $n \leq 1,024$ and 149 γ values at $n = pq$ for $n \leq 512$.) The relative errors range from about -10.5% to -5.7%. For a quadratic least-squares fit of the values, SSE=0.0001112, R-squared=0.995, and RMSE=0.000888. See Figure (230) for a plot of the slopes of the linear least-squares fits of the $\sum_{i|n} \kappa_1(n/i)$ values at $n = 2p$ for n less than or equal to 1,024, 2,048, 4,096,..., 4,194,304. For a quadratic least-squares fit of the values, SSE=0.005733, R-squared=0.9983, and RMSE=0.02283. See Figure (231) for a plot of the intercepts of the linear least-squares fits of the $\sum_{i|n} \kappa_1(n/i)$ values. The values rapidly increase. See Figure (232) for a plot of the ratios of the expected and actual $\sum_{i|n} \kappa_1(n/i)$ values. The values slowly approach 1. See Figure (233) for a plot of the scaling factors of the $\sum_{i|n} \kappa_2(n/i)$ values. For a quadratic least-squares fit of the values, SSE=0.0001185, R-squared=0.9931, and RMSE=0.003442. The numbers of estimated γ values at $n = pq$ for n equal to 2,048, 4,096, 8,192, ..., 4,194,304 are 282, 531, 1,056, 1,999, 3,907, 7,536, 14,577, 28,268, 54,730, 106,408, 206,364, and 401,442 respectively. The ranges of the relative errors of the estimated γ values are (-5.94%, -2.98%), (-4.01%, -1.97%), (-2.82%, -1.32%), (-1.67%, -0.55%), (-1.61%, -0.74%), (-1.05%, -0.35%), (-0.89%, -0.31%), (-0.79%, -0.30%), (-0.68%, -0.25%), (-0.70%, -0.32%), (-0.59%, -0.26%), and (-0.58%, -0.28%) respectively. See Figure (234) for a plot of the relative errors for $n = 4,194,304$. For a quadratic least-squares fit of the values, SSE=$4.109\cdot10^{-5}$, R-squared=0.9999, and RMSE=$1.012\cdot10^{-5}$.

For a linear least-squares fit of the 21 $\sum_{i|n} \kappa_1(n/i)$ values at $n = 6p$ ($p \neq 2$, $p \neq 3$) and for $n \leq 512$, the slope is 80.31 with a 95% confidence interval of (78.66, 81.96), the intercept is 259 with a 95% confidence interval of (238.3, 279.7), SSE=9.080, R-squared=0.9982, and RMSE=21.86. For $n \leq 1,024$, there are 37 $\sum_{i|n} \kappa_1(n/i)$ values at $n = 6p$, so the estimated value of $\sum_{i|n} \kappa_1(n/i)$ at the largest $n = 6p$ value is 3230.47 (80.31·37+259). The actual value is 3269.383744. The scaling factor for $\sum_{i|n} \kappa_2(n/i)$ values at $n = pqr$ and for $n \leq 1,024$ is then 1.72542247 ((3230.47-346.893092)/(1921.377233-250.147742)). 346.893092 is the $\sum_{i|n} \kappa_1(n/i)$ value at $n = 30$, 1921.377233 is the $\sum_{i|n} \kappa_2(n/i)$ value at the largest $n = 6p$ value for $n \leq 1,024$, and 250.147742 is the $\sum_{i|n} \kappa_2(n/i)$ value at $n = 30$. Möbius inversion is then done using this scaling factor and normalization value for all $\sum_{i|n} \kappa_2(n/i)$ values for $n \leq 1,024$ and the estimated γ values at $n = pqr$ are extracted. See Figure (235) for a plot of the relative errors for the last 80 ($142 - 62$) estimated values. (There are 142 γ values at $n = pqr$ for $n \leq 1,024$ and 62 γ values at $n = pqr$ for $n \leq 512$.) The relative errors range from about 20.31% to 13.16%. For a quadratic least-squares fit of the values, SSE=0.000192, R-squared=0.994, and RMSE=0.001579. See Figure (236) for a plot of the ratios of the expected and actual $\sum_{i|n} \kappa_1(n/i)$ values at $n = 6p$ for n less than or equal to 1,024, 2,048, 4,096,..., 4,194,304. The values slowly approach 1. See Figure (237) for a plot of the scaling factors of the $\sum_{i|n} \kappa_2(n/i)$ values. For a quadratic least-squares fit of the values, SSE=0.0005699, R-squared=0.9822, and RMSE=0.007549. The numbers of estimated γ values at $n = pqr$ for n equal to 2,048, 4,096, 8,192, ..., 4,194,304 are 168, 377, 749, 1,604, 3,223, 6,612, 13,404, 27,050, 54,505, 109,245, 219,398, and 438,329 respectively. The ranges of the relative errors of the estimated γ values are (12.38%, 8.41%), (6.11%, 3.99%), (4.23%, 3.34%), (2.07%, 1.81%), (1.33%, 1.22%), (0.75%, 0.47%), (0.74%, 0.39%), (0.40%, 0.0299%), (0.76%, 0.40%), (0.81%, 0.47%), (0.21%, −0.10%), and (−0.0574%, 0.23%) respectively. See Figure (238) for a plot of the relative errors for $n = 4,194,304$. For a quadratic least-squares fit of the values, SSE=$4.887 \cdot 10^{-5}$, R-squared=0.9998, and RMSE=$1.056 \cdot 10^{-5}$.

For a linear least-squares fit of the 14 $\sum_{i|n} \kappa_1(n/i)$ values at $n = 72p$ ($p \neq 2$, $p \neq 3$) and for $n \leq 4,000$, the slope is 910.7 with a 95% confidence interval of (874.9, 946.5), the intercept is 1,816 with a 95% confidence interval of (1,511, 2,121), SSE=$7.383 \cdot 10^5$, R-squared=0.9961, and RMSE=248. For $n \leq 20,000$, there are 57 $\sum_{i|n} \kappa_1(n/i)$ values at $n = 72p$, so the estimated value of $\sum_{i|n} \kappa_1(n/i)$ at the largest $n = 72p$ value is 53725.9 (910.7·57+1816). The actual value is 56906.517248. The scaling factor for $\sum_{i|n} \kappa_5(n/i)$ values at $n = p^3q^2r$ and for $n \leq 20,000$ is then 3.6791295316 ((53725.9-2962.398030)/(14899.032929-1101.339335)). 2962.398030 is the $\sum_{i|n} \kappa_1(n/i)$ value at $n = 360$, 14899.032929 is the $\sum_{i|n} \kappa_5(n/i)$ value at the largest $n = 72p$ value for $n \leq 20,000$, and 1101.339335 is the $\sum_{i|n} \kappa_5(n/i)$ value at $n = 360$. Möbius inversion is then done using this scaling factor and normalization value for all $\sum_{i|n} \kappa_5(n/i)$ values for $n \leq 20,000$ and the estimated γ values at $n = p^3q^2r$ are extracted. See Figure (239) for a plot of the relative errors for the last 153 $(191 - 38)$ estimated values. (There are 191 γ values at $n = p^3q^2r$ for $n \leq 20,000$ and 38 γ values at $n = p^3q^2r$ for $n \leq 5,000$.) The relative errors range from about 3.29% to 8.11%. For a quadratic least-squares fit of the values, SSE=$9.702 \cdot 10^{-5}$, R-squared=0.9964, and RMSE=0.0008042. The numbers of estimated γ values at $n = p^3q^2r$ for n equal to 100,000, 500,000, and 2,500,000 are 629, 2,541, and 10,538 respectively. The ranges of the relative errors of the estimated γ values are (3.88%, 7.28%), (2.92%, 5.46%), and (2.02%, 3.98%) respectively. See Figure (240) for a plot of the relative errors for $n = 2,500,000$. For a quadratic least-squares fit of the values, SSE=0.0008841, R-squared=0.997, and RMSE=0.0002897.

See Figure (241) for a plot of the 23 $\sum_{i|n} \kappa_1(n/i)$ values at $n = 4p^2$ ($p \neq 2$) for $n \leq 32,768$. For a quadratic least-squares fit of the values, $p_1 = 99.58$ with a 95% confidence interval of (92.1, 107.1), $p_2 = -124.7$ with a 95% confidence interval of (−309.8, 60.37), $p_3 = 497.9$ with a 95% confidence interval of (−466.3, 1462), SSE=$9.129 \cdot 10^6$, R-squared=0.9984, and RMSE=675.6. For $n \leq 65,536$, there are 30 $\sum_{i|n} \kappa_1(n/i)$ values at $n = 4p^2$, so the estimated value of $\sum_{i|n} \kappa_1(n/i)$ at the largest $n = 4p^2$ value is 94754.225139. The actual value is 86378.9. The scaling factor for $\sum_{i|n} \kappa_2(n/i)$ values at $n = p^2q^2$ and for $n \leq 65,536$ is then 1.83876902276 ((94754.225139-419.017454)/(51601.889447-298.432389)). 419.017454 is the $\sum_{i|n} \kappa_1(n/i)$ value at $n = 36$, 51601.889447 is the $\sum_{i|n} \kappa_2(n/i)$ value at the largest $n = 4p^2$ value for $n \leq 65,536$, and 298.432389 is the $\sum_{i|n} \kappa_2(n/i)$ value at $n = 36$. Mobius inversion is then done using this scaling factor and normalization value for all $\sum_{i|n} \kappa_2(n/i)$ values for $n \leq 65,536$ and the estimated γ values at $n = p^2q^2$ are extracted. See Figure (242) for a plot of the relative errors of the last 24 $(76 - 52)$ estimated values. (There are 76 γ values at $n = p^2q^2$ for $n \leq 65,536$ and 52 γ values at $n = p^2q^2$ for $n \leq 32,768$.) The relative errors range from about 8.75% to 9.26%. The numbers of estimated γ values at $n = p^2q^2$ for n equal to 131,072, 262,144, 524,288, ..., 4,194,304 are 31, 42, 62, 82, 120, and 162 respectively. The ranges of the relative errors of the estimated γ values are (5.05%, 5.53%), (1.88%, 2.32%), (3.29%, 3.68%), (0.11%, 0.46%), (3.68%, 3.99%), and (3.21%, 3.50%) respectively. See Figure (243) for a plot of the relative errors for $n = 4,194,304$. For a quadratic least-squares fit of the values, SSE=$1.357 \cdot 10^{-7}$, R-squared=0.9987, and RMSE=$2.922 \cdot 10^{15}$.

See Figure (244) for a plot of the 14 $\sum_{i|n} \kappa_1(n/i)$ values at $n = 8p^3$ $(p \neq 2)$ for $n \leq 1,048,576$. For a cubic least-squares fit of the values, $p_1 = 239.9$ with a 95% confidence interval of $(-108.3,\ 588.1)$, $p_2 = 3575$ with a 95% confidence interval of $(-4352,\ 11500)$, $p_3 = -25190$ with a 95% confidence interval of $(-77270,\ 26900)$, $p_4 = 36070$ with a 95% confidence interval of $(-57360,\ 129500)$, SSE=$8.549 \cdot 10^{10}$, R-squared=0.9945, and RMSE=$2.924 \cdot 10^4$. For $n \leq 2,097,152$, there are 17 $\sum_{i|n} \kappa_1(n/i)$ values at $n = 8p^3$, so the estimated value of $\sum_{i|n} \kappa_1(n/i)$ at the largest $n = 8p^3$ value is 1819643.7. The actual value is 2087770.661312. The scaling factor for $\sum_{i|n} \kappa_2(n/i)$ values at $n = p^3q^3$ and for $n \leq 2,907,152$ is then 1.6367999527 ((1819643.7-1651.162704)/(1111760.929595-1061.644516)). 1651.162704 is the $\sum_{i|n} \kappa_1(n/i)$ value at $n = 216$, 1111760.9299595 is the $\sum_{i|n} \kappa_2(n/i)$ value at the largest $n = 8p^3$ value for $n \leq 2,097,152$, and 1061.644516 is the $\sum_{i|n} \kappa_2(n/i)$ value at $n = 216$. Mobius inversion is then done using this scaling factor and normalization value for all $\sum_{i|n} \kappa_2(n/i)$ values for $n \leq 2,097,152$ and the estimated γ values at $n = p^3q^3$ are extracted. See Figure (245) for a plot of the relative errors of the last 7 $(37 - 30)$ estimated values. (There are 37 γ values at $n = p^3q^3$ for $n \leq 2,097,152$ and 30 γ values at $n = p^3q^3$ for $n \leq 1,048,576$.) The relative errors range from about 12.99% to 13.17%. For a quadratic least-squares fit of the values, R-squared=0.9916. The number of estimated γ values at $n = p^3q^3$ for n equal to 4,194,304 is 12. The range of the relative errors of the estimated γ values is (−2.58%, −2.29%). See Figure (246) for a plot of the relative errors for $n = 4,194,304$. For a quadratic least-squares fit of the values, R-squared=0.9686.

12 Dirichlet Products of Eigenvalues of Gaussian-Random Hermitian Matrices

Let $\lambda_1(1)$, $\lambda_1(2)$, $\lambda_1(3)$, ..., $\lambda_1(d)$ denote e_1, e_2, e_3, ..., e_d and let $\lambda_m(n)$, $m = 2$, 3, 4, ..., denote these values and $m-1$ values that have been linearly interpolated between successive values. In the following, d is set to 4,500. See Figure (247) for plots of $\sum_{i|n} \lambda_1(n/i)$ and $\sum_{i|n} \lambda_4(n/i)$ at n values that are of the form pqr for n less than or equal to 1,000. Note that only 250 e values are used to compute $\sum_{i|n} \lambda_4(n/i)$. See Figure (248) for plots of the normalized and scaled values. The curves are normalized by subtracting the first element of each curve. The $\sum_{i|n} \lambda_4(n/i)$ values are then scaled by multiplying them by the ratio of the last elements of the normalized curves. The resulting curves are almost the same. The normalization value of the $\sum_{i|n} \lambda_4(n/i)$ values is -1505.145075 and the scaling factor is 2.5989836788. Mobius inversion is then done using this scaling factor and normalization value for all $\sum_{i|n} \lambda_4(n/i)$ values for $n \leq 1,000$ and the estimated e values at $n = pqr$ are extracted. See Figure (249) for a plot of the estimated and actual e values (an additional normalization of -1801.707412 of the estimated values is required to set the first value to e_1). See Figure (250) for a plot of the relative errors of the estimated values. The values range from 0.16% to -1.74%. Of course, the last $\sum_{i|n} \lambda_1(n/i)$ value would have to be predetermined to compute the scaling factor.

See Cox [6] for the software used to determine the above results.

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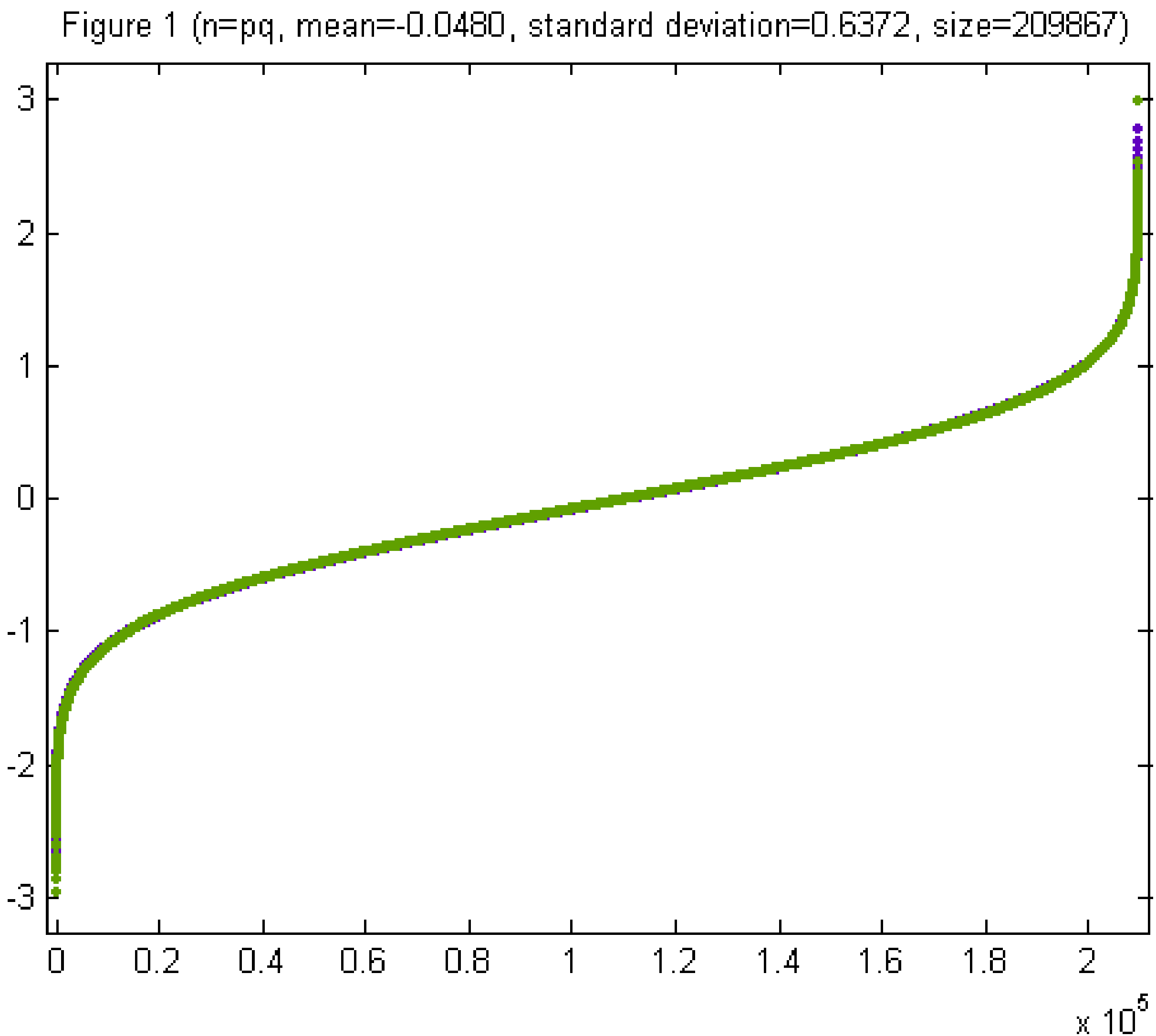


Figure 1 (n=pq, mean=-0.0480, standard deviation=0.6372, size=209867)
3
2
1
0
-1
-2
-3
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2
x 10^5

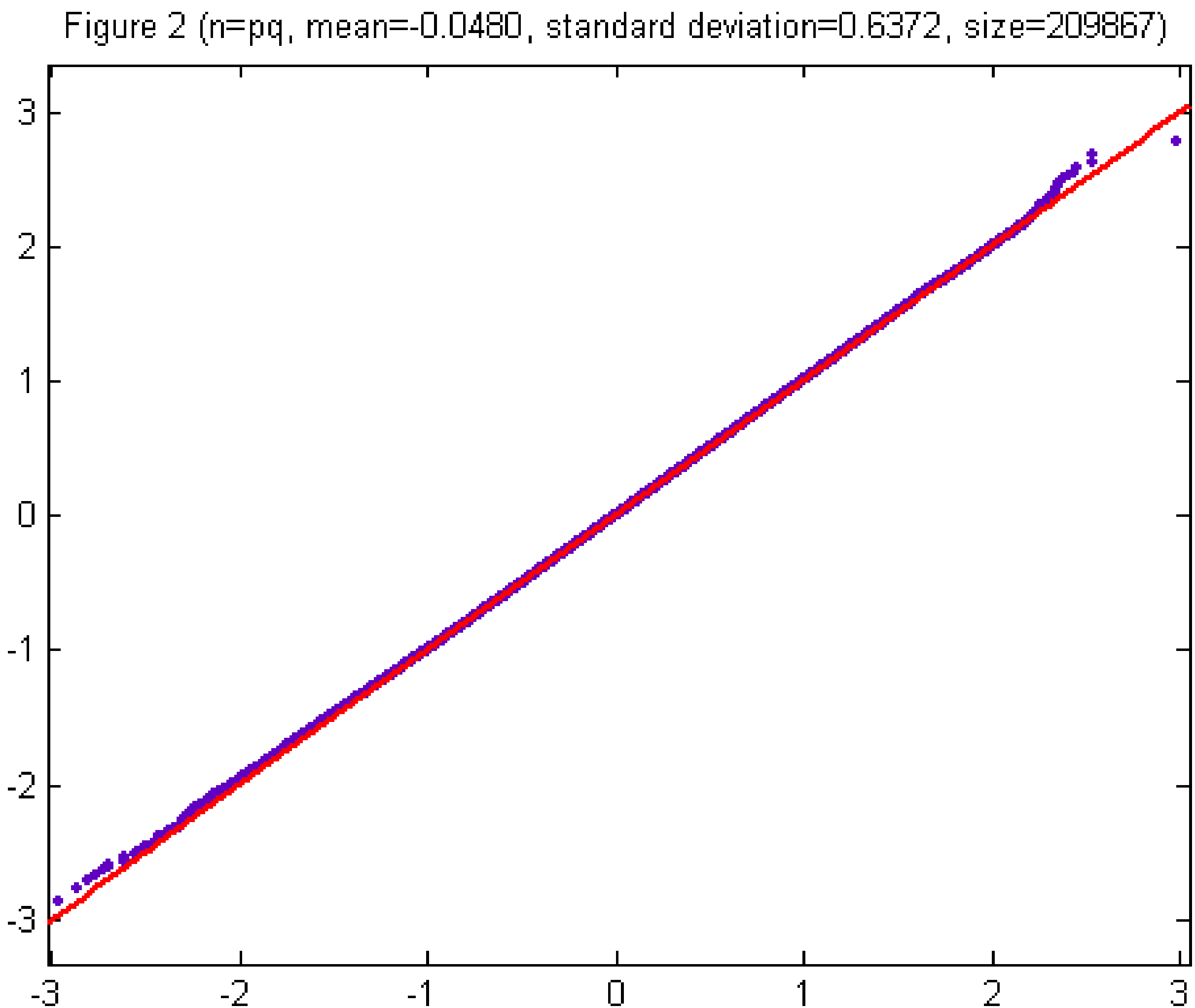


Figure 2 (n=pq, mean=-0.0480, standard deviation=0.6372, size=209867)
3
2
1
0
-1
-2
-3
-3
-2
-1
0
1
2
3

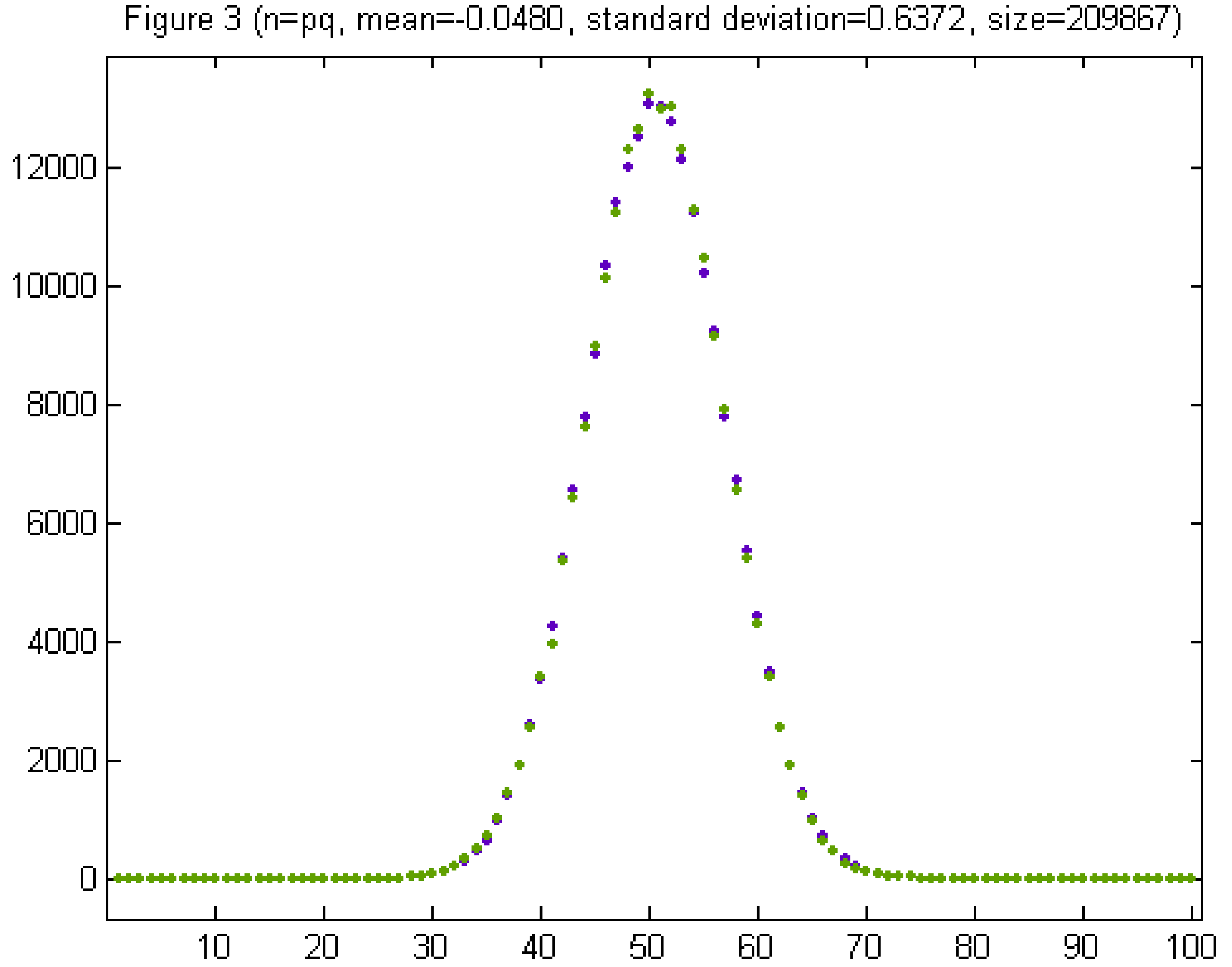


Figure 3 (n=pq, mean=-0.0480, standard deviation=0.6372, size=209867)
0
2000
4000
6000
8000
10000
12000
10
20
30
40
50
60
70
80
90
100

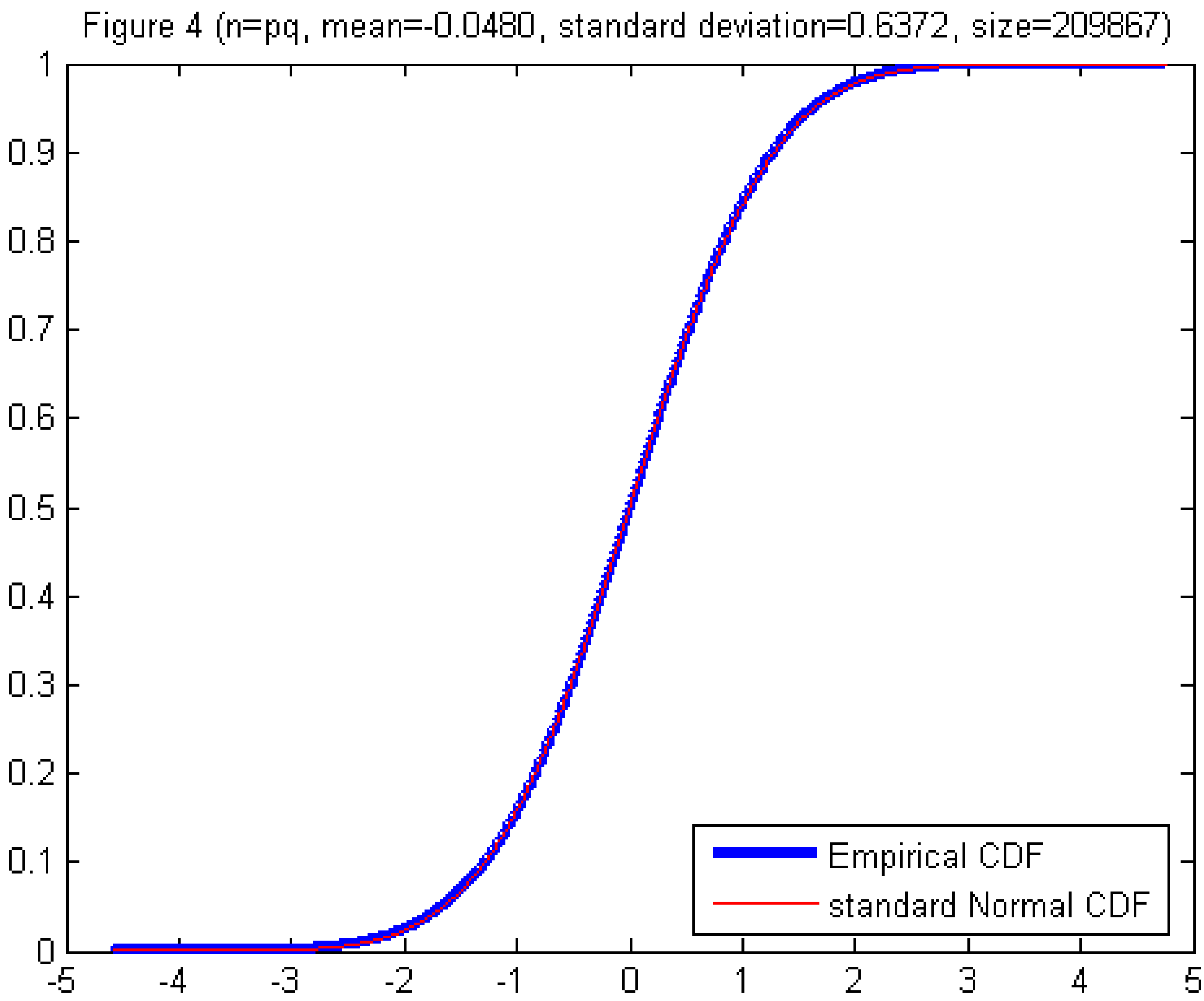


Figure 4 (n=pq, mean=-0.0480, standard deviation=0.6372, size=209867)
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
-5
-4
-3
-2
-1
0
1
2
3
4
5
Empirical CDF
standard Normal CDF

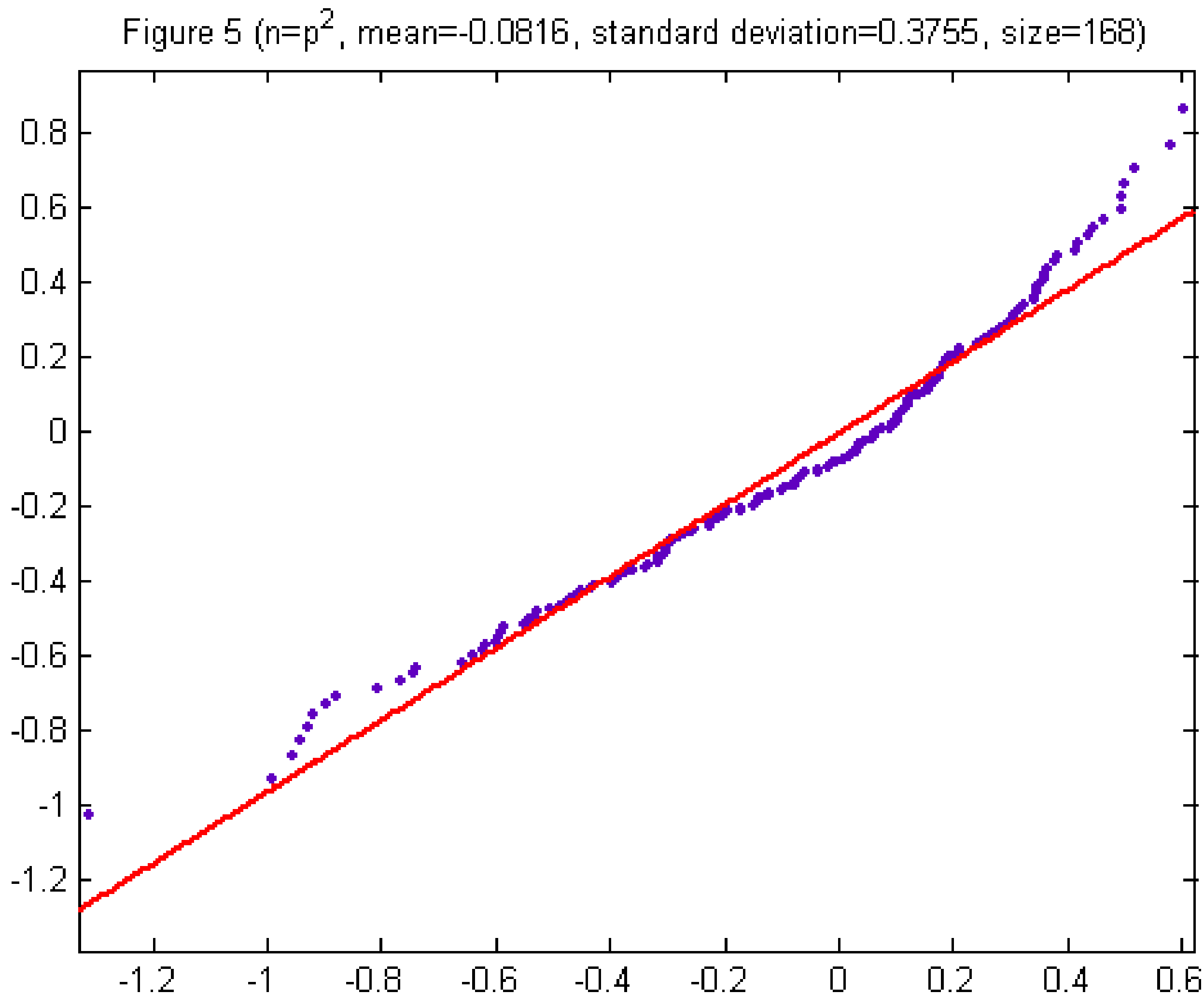
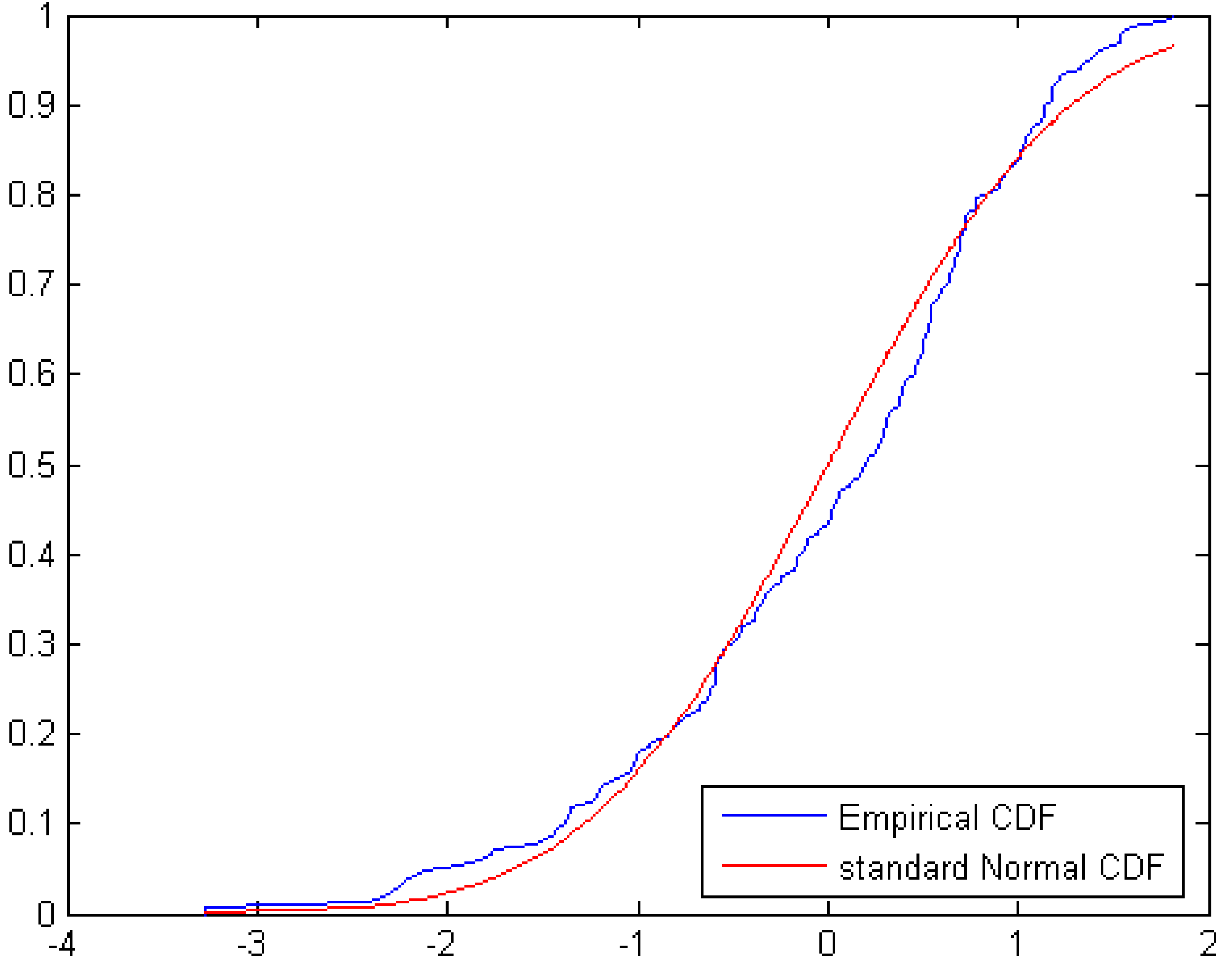


Figure 5 (n=p2, mean=-0.0816, standard deviation=0.3755, size=168)
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
-1.2
-1.2
-1
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
0.6

Figure 6 ($n=p^2$, mean=-0.0816, standard deviation=0.3755, size=168)



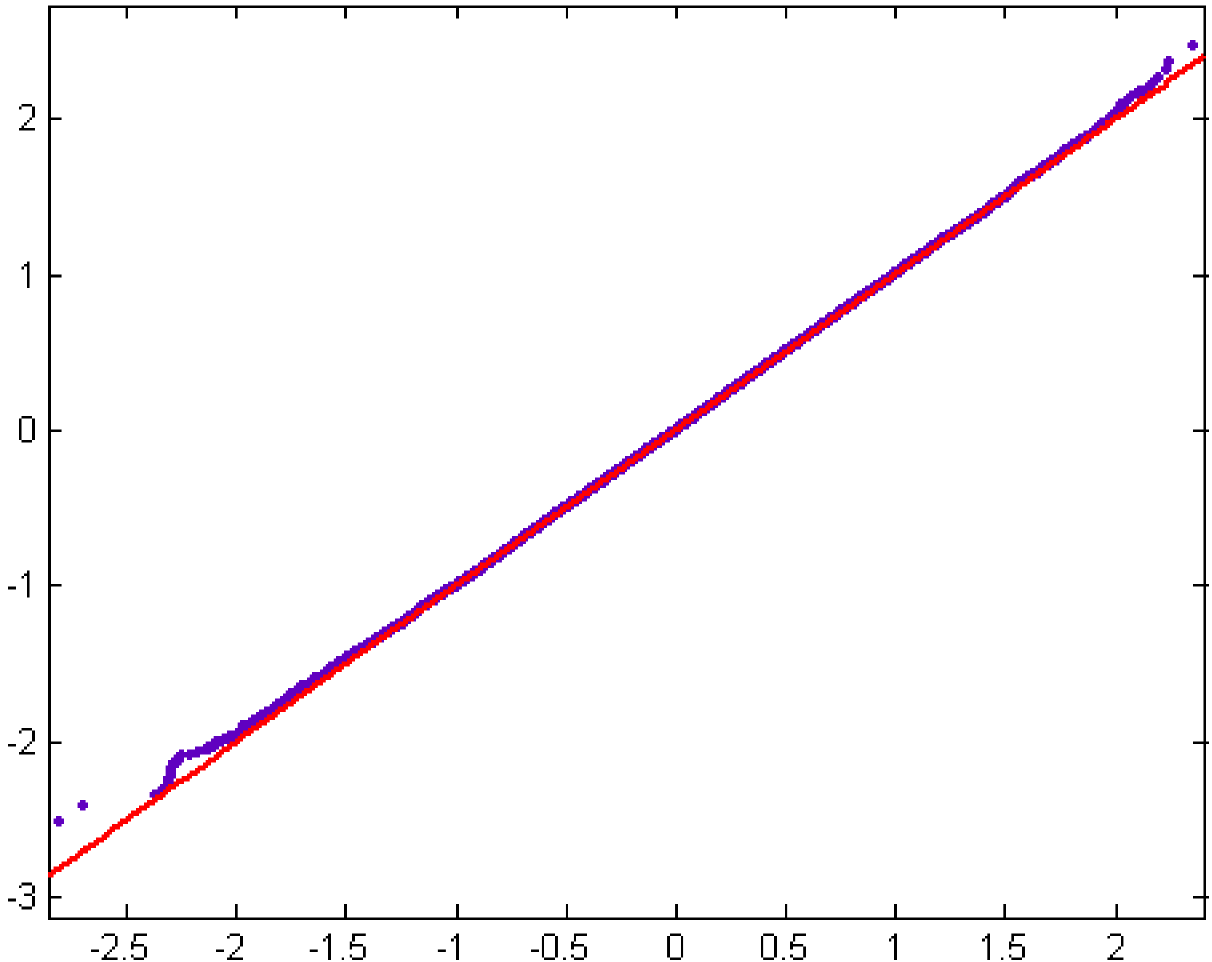


Figure 7 ($n=p^2q$, mean=-0.0239, standard deviation=0.6103, size=43864)

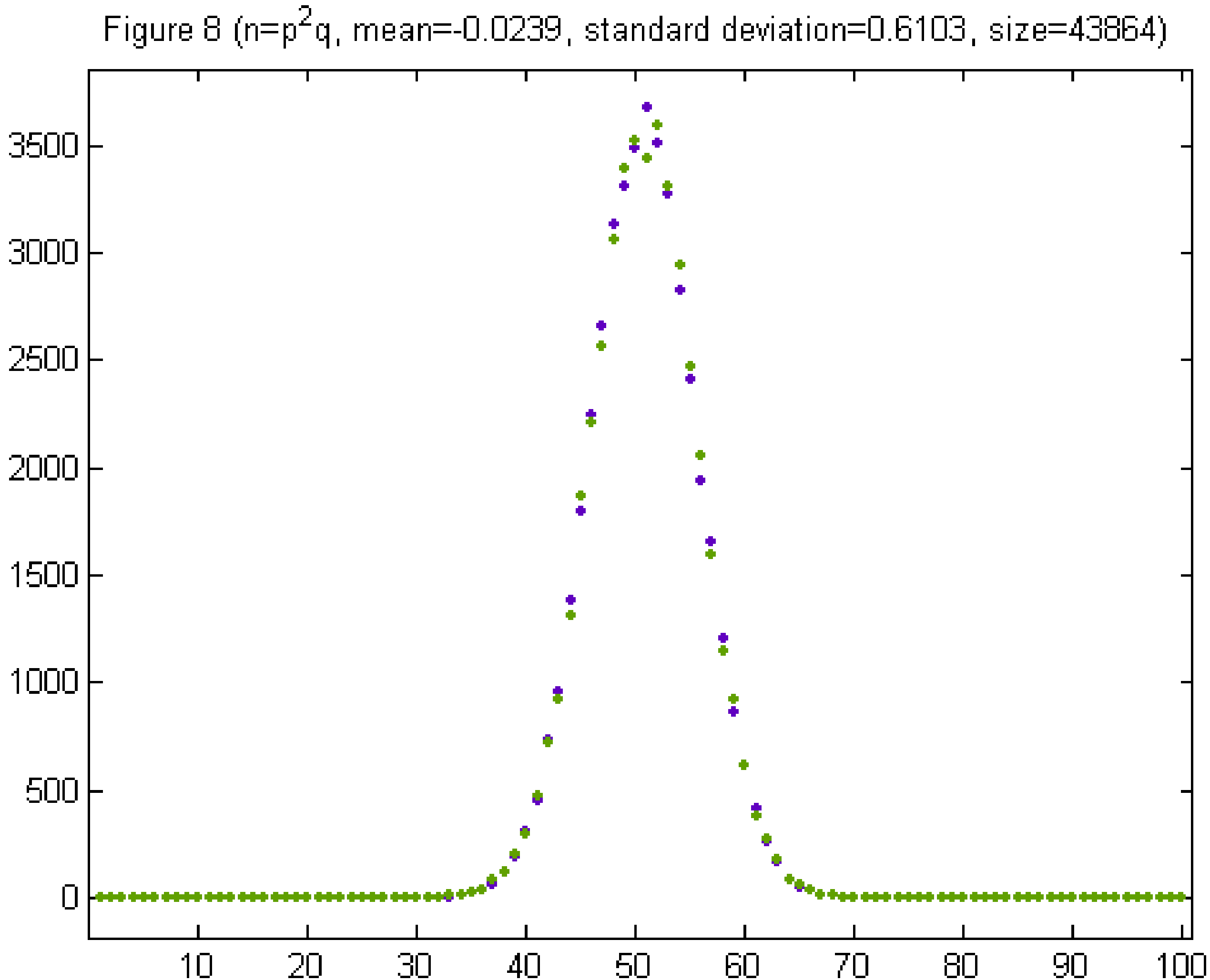


Figure 8 (n=p²q, mean=-0.0239, standard deviation=0.6103, size=43864)
3500
3000
2500
2000
1500
1000
500
0
10
20
30
40
50
60
70
80
90
100

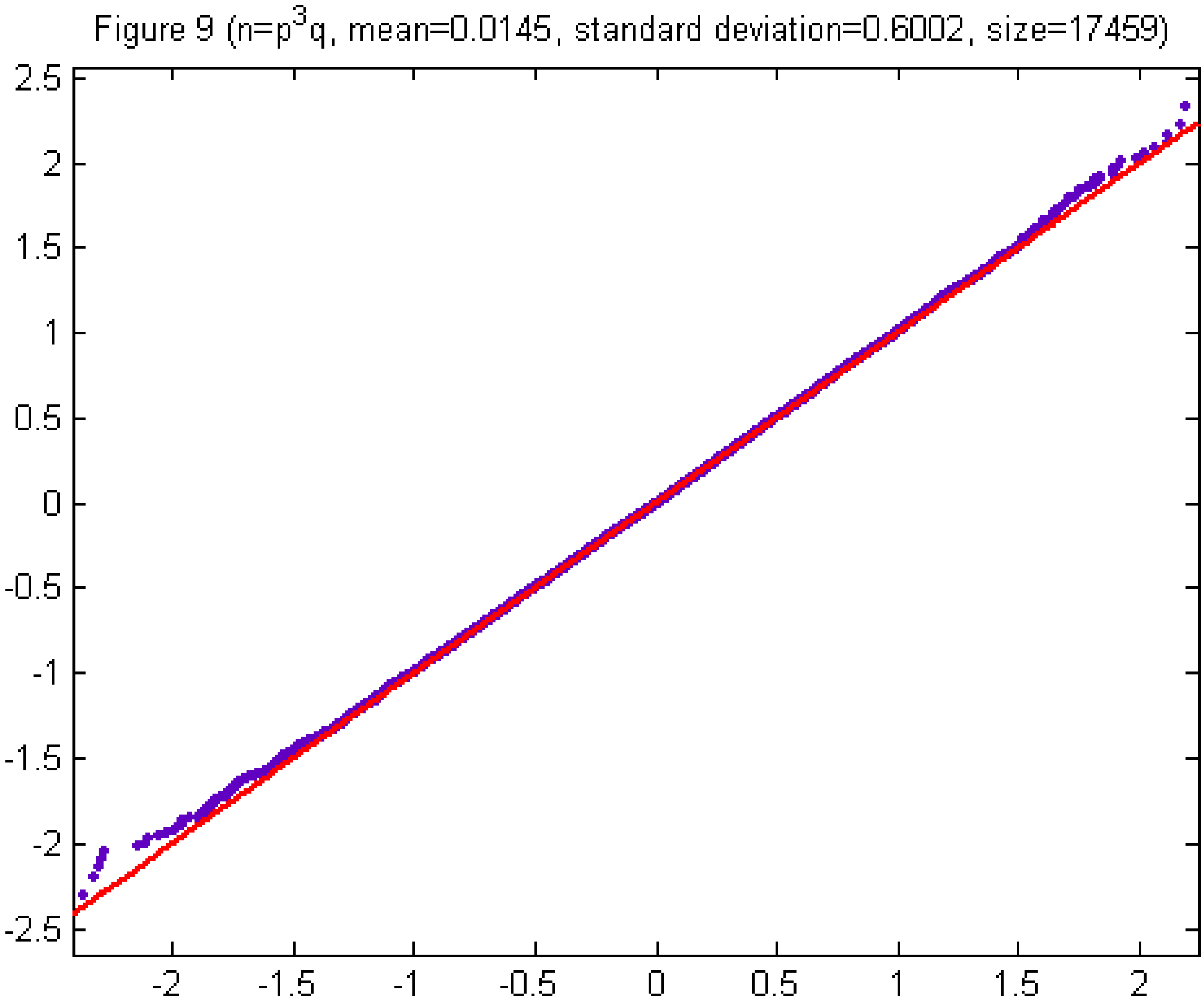


Figure 9 (n=p^3q, mean=0.0145, standard deviation=0.6002, size=17459)
2.5
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
-2.5
-2
-1.5
-1
-0.5
0
0.5
1
1.5
2

Figure 10 ($n=p^3q$, mean=0.0145, standard deviation=0.6002, size=17459)

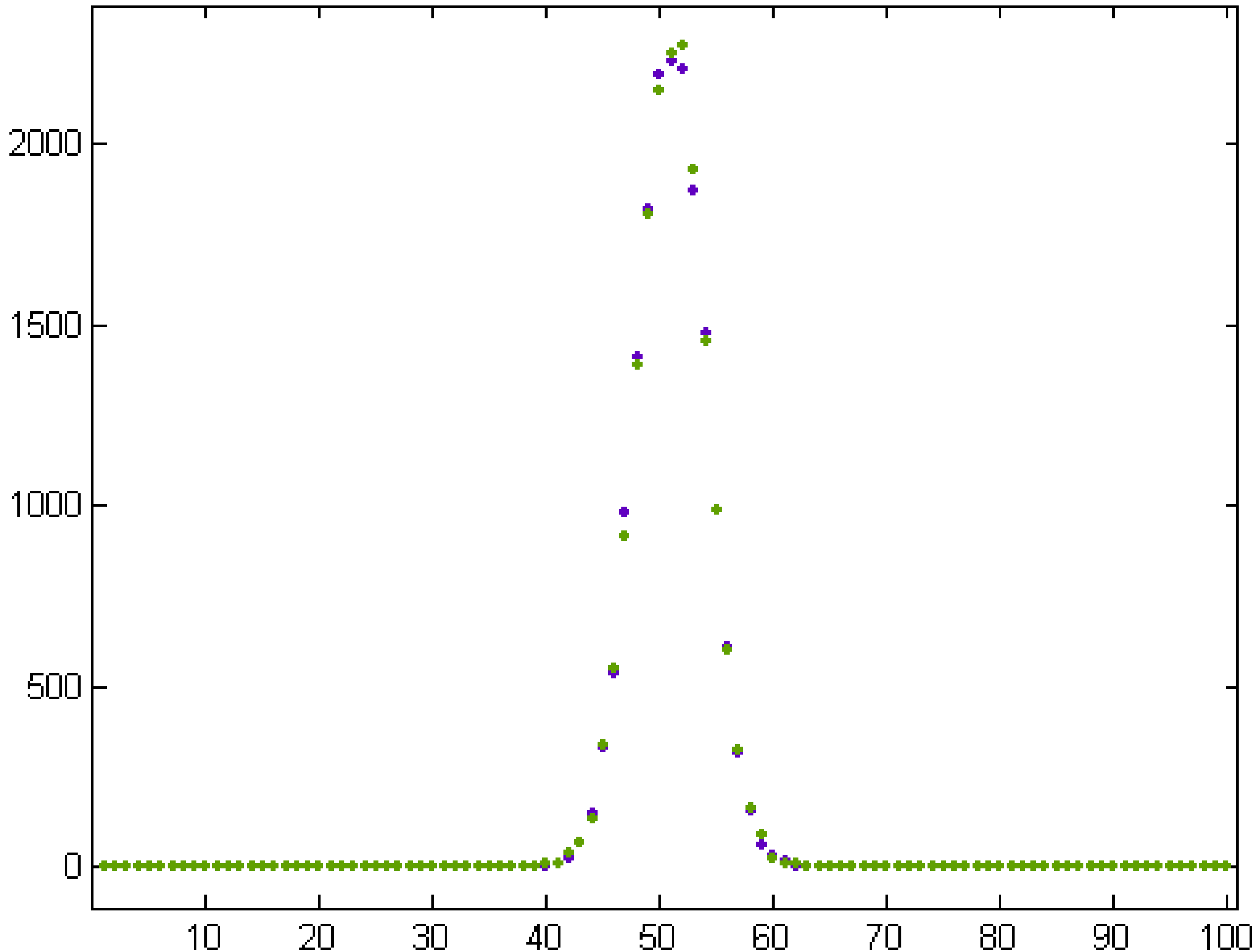
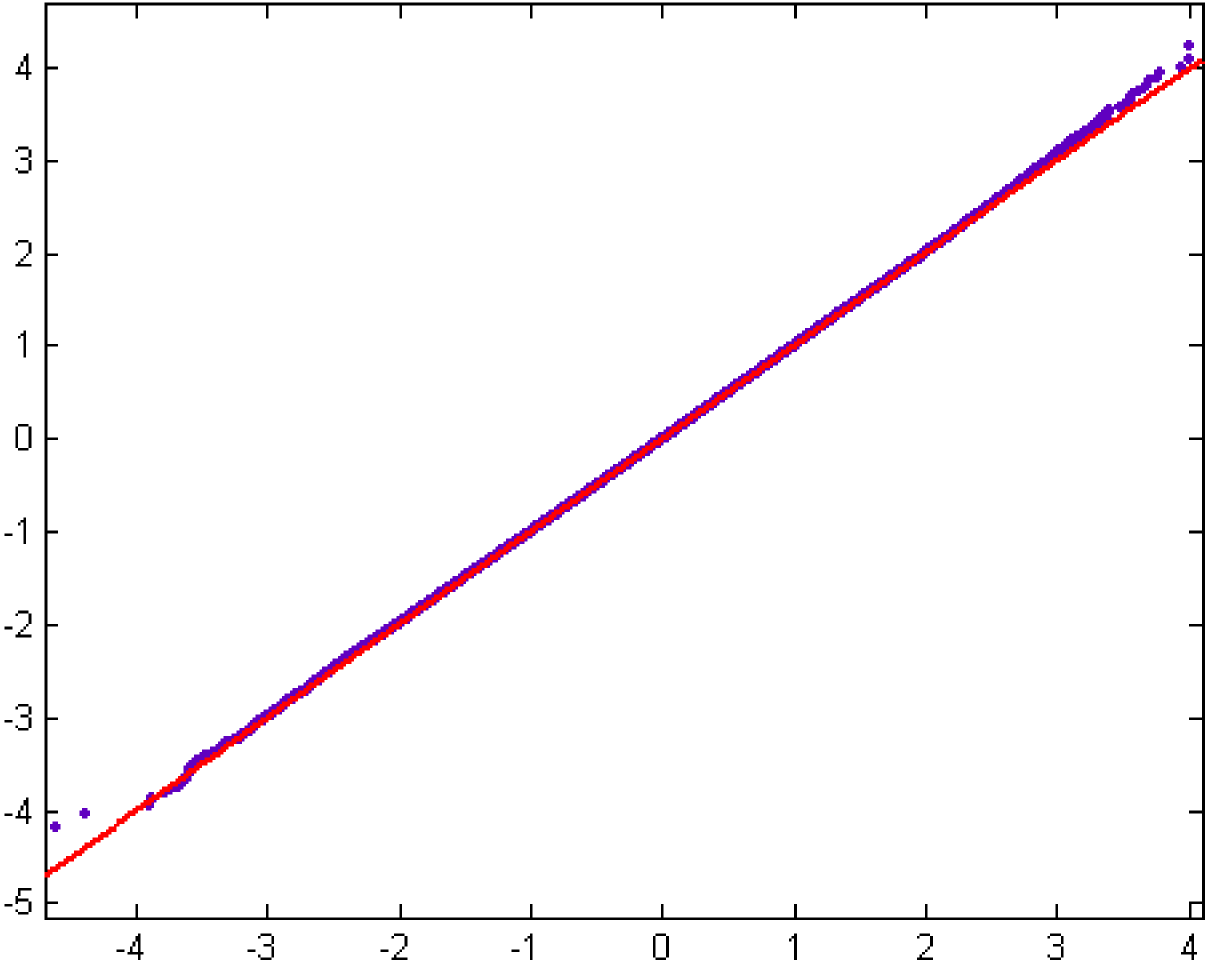


Figure 11 (n=pqr, mean=0.0158, standard deviation=0.9521, size=206964)



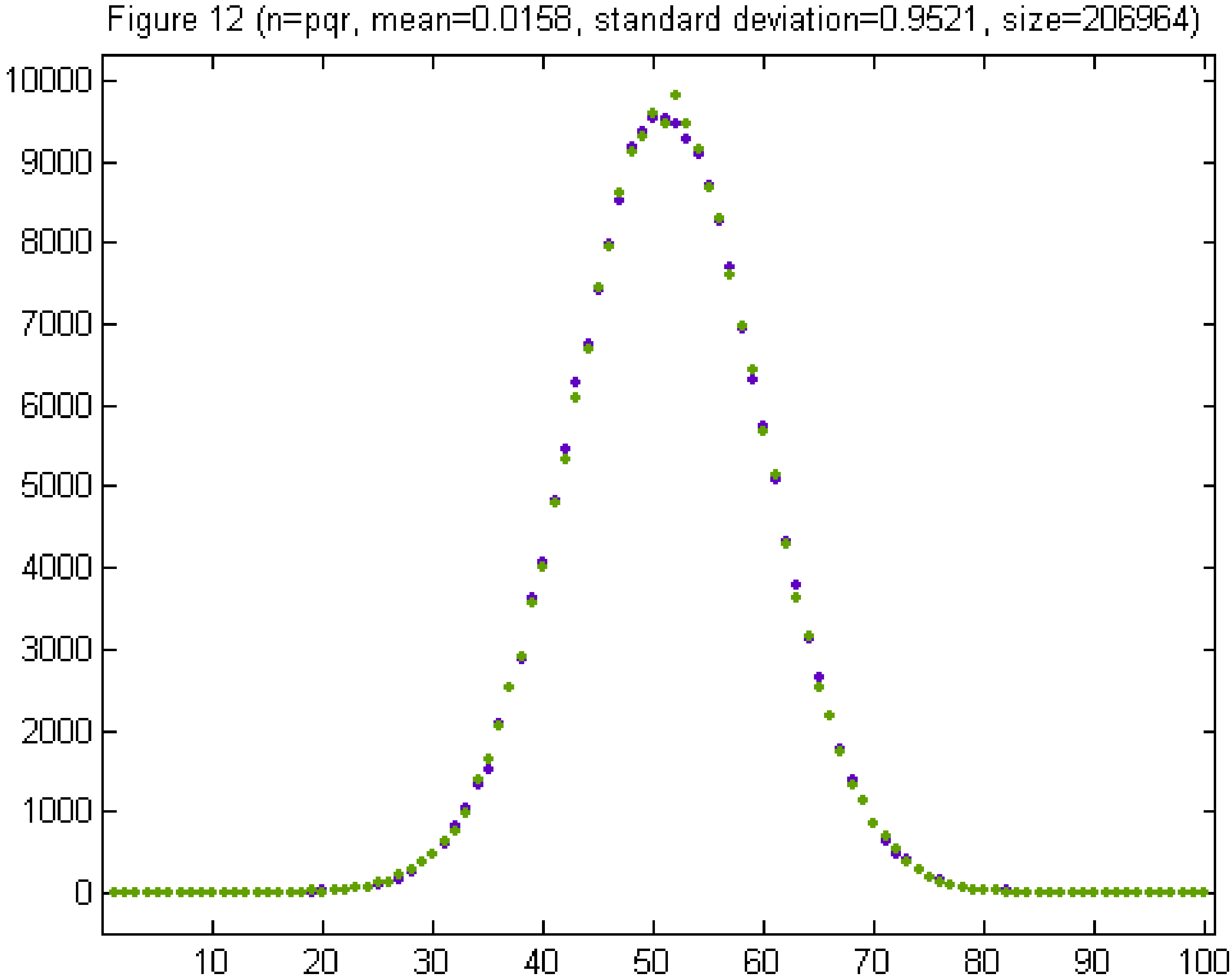


Figure 12 (n=pqr, mean=0.0158, standard deviation=0.9521, size=206964)
10000
9000
8000
7000
6000
5000
4000
3000
2000
1000
0
10
20
30
40
50
60
70
80
90
100

Figure 13 ($n=p^4q$, mean=0.1025, standard deviation=0.8221, size=8114)

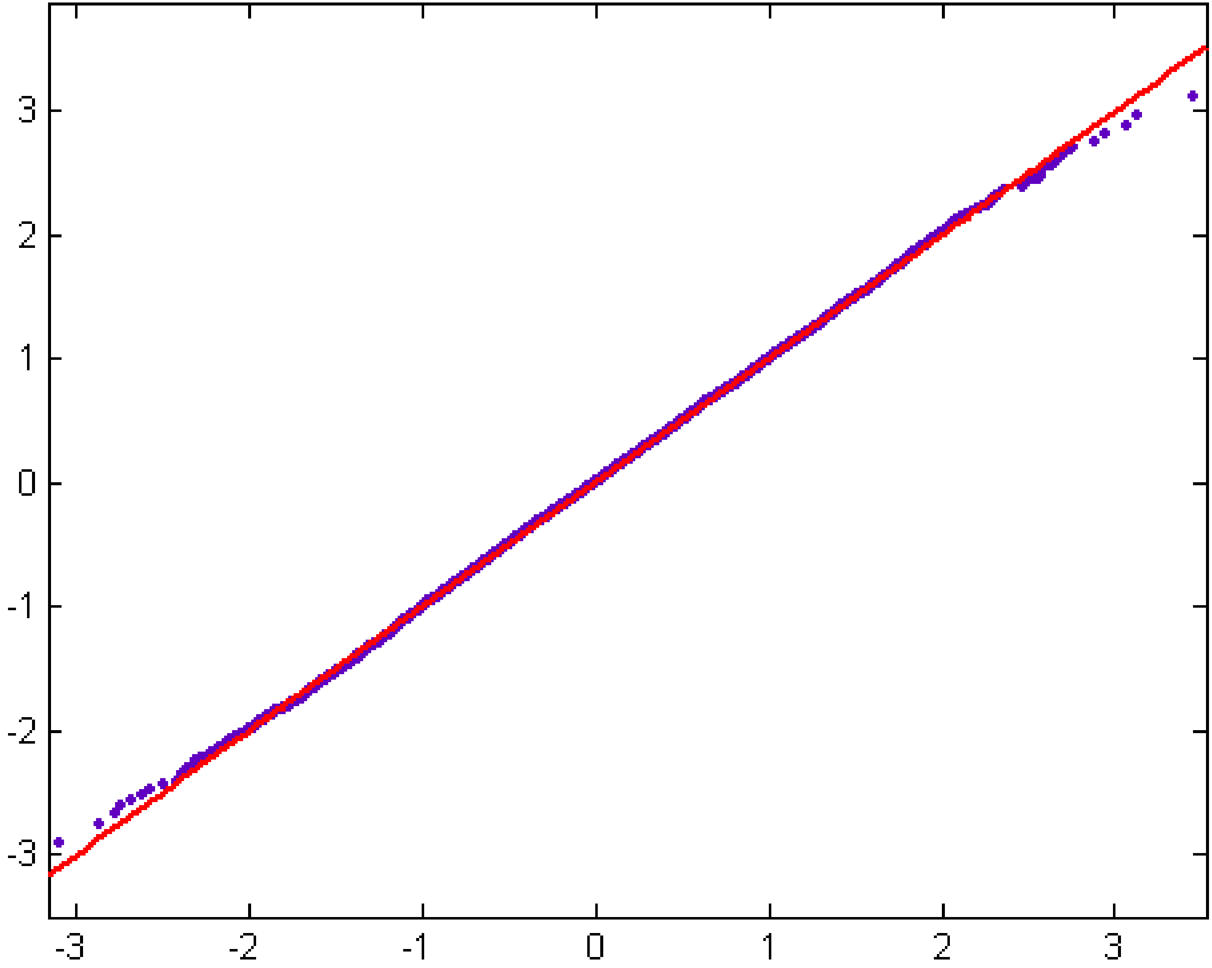
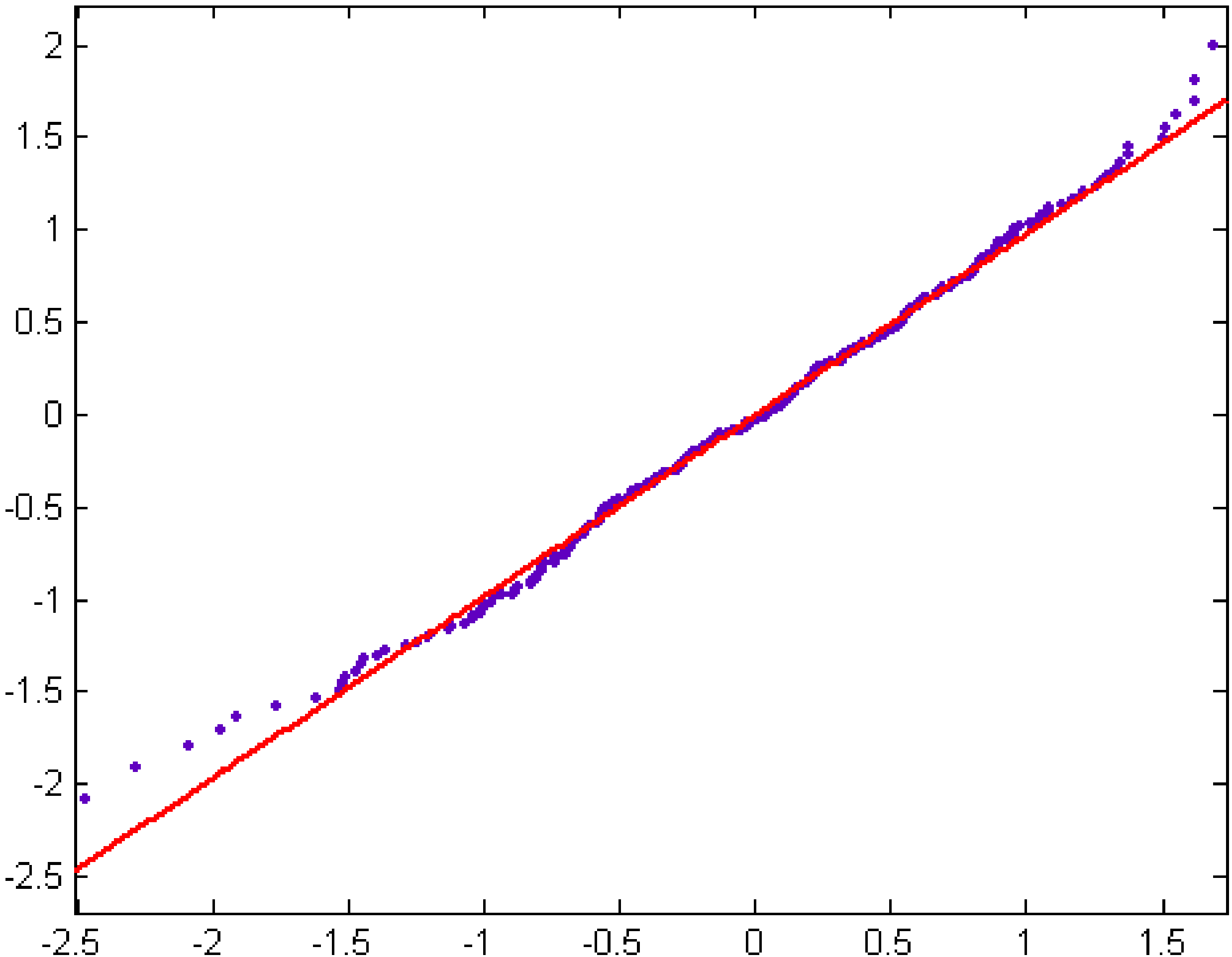


Figure 14 ($n=p^2q^2$, mean=-0.0478, standard deviation=0.7565, size=287)



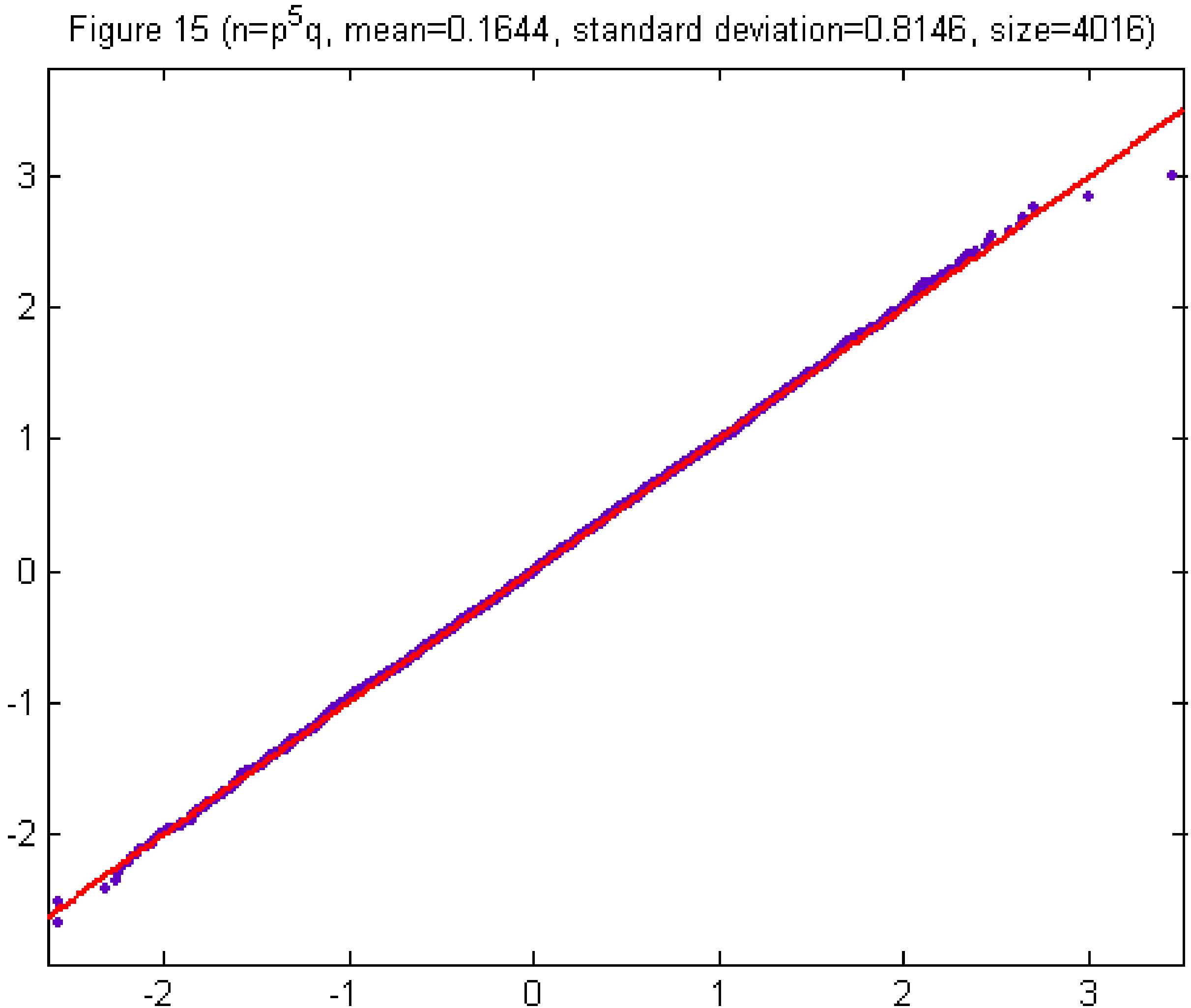
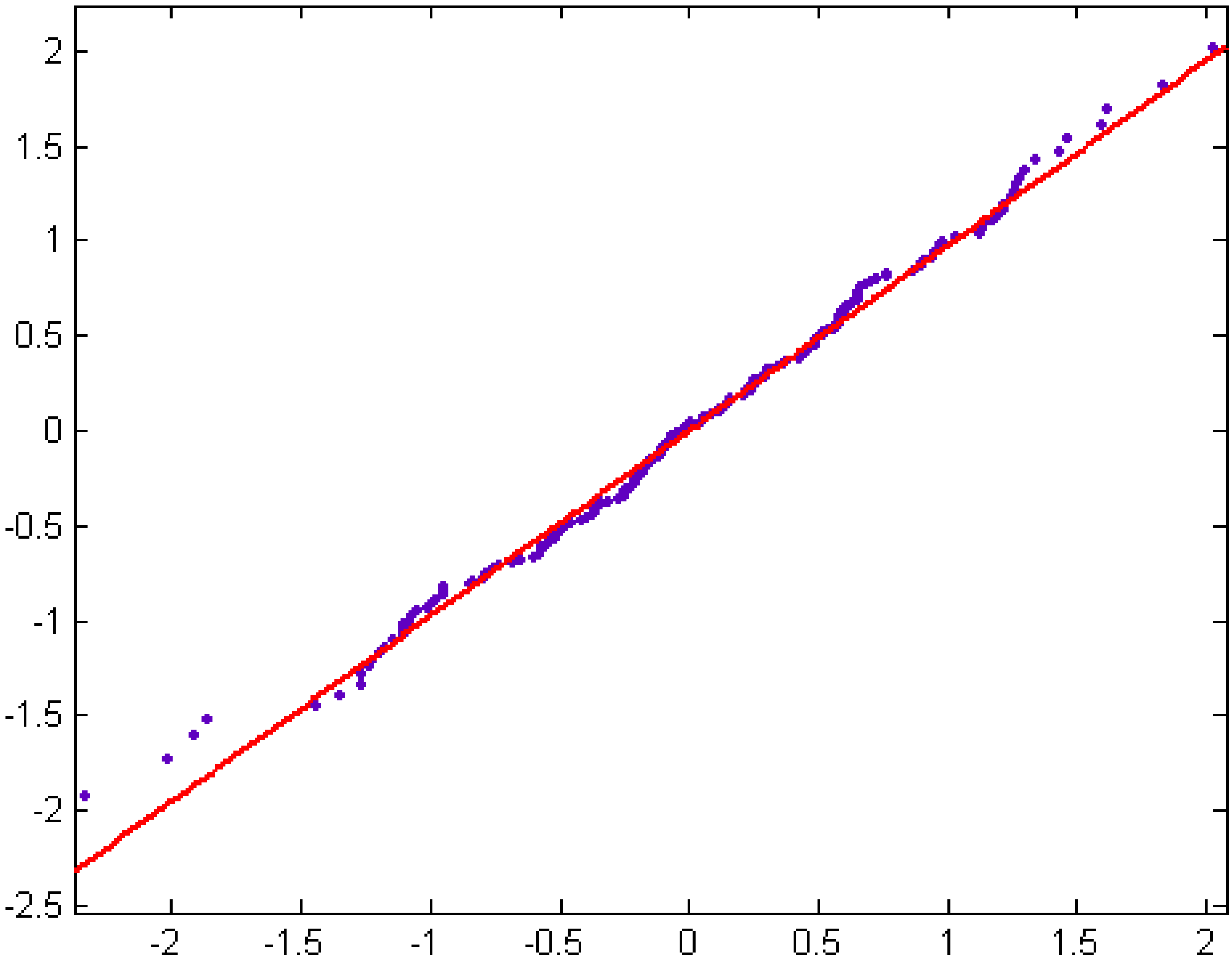


Figure 15 (n=p5q, mean=0.1644, standard deviation=0.8146, size=4016)
3
2
1
0
-1
-2
-2
-1
0
1
2
3

Figure 16 ($n=p^3q^2$, mean=0.0417, standard deviation=0.7650, size=196)



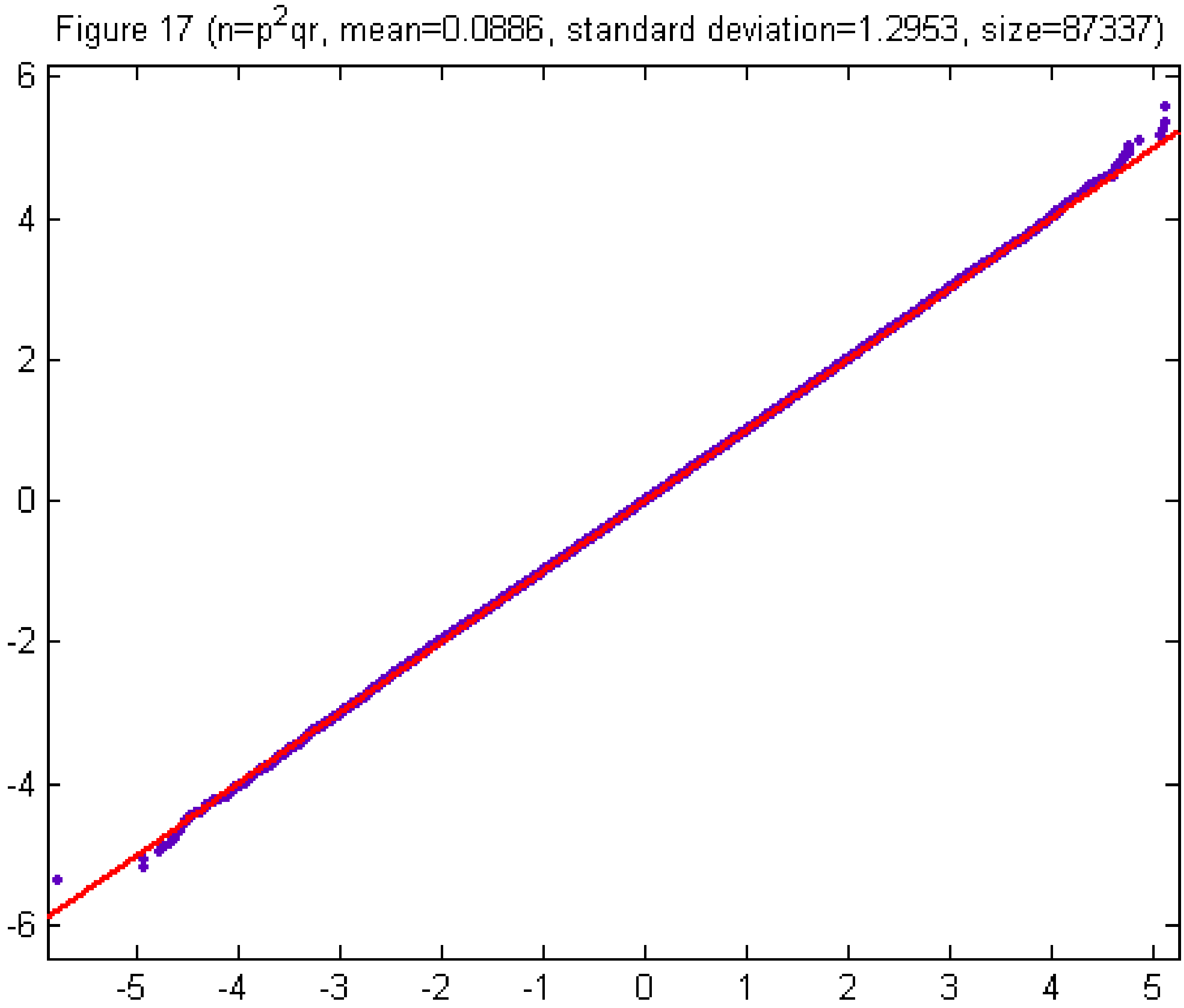
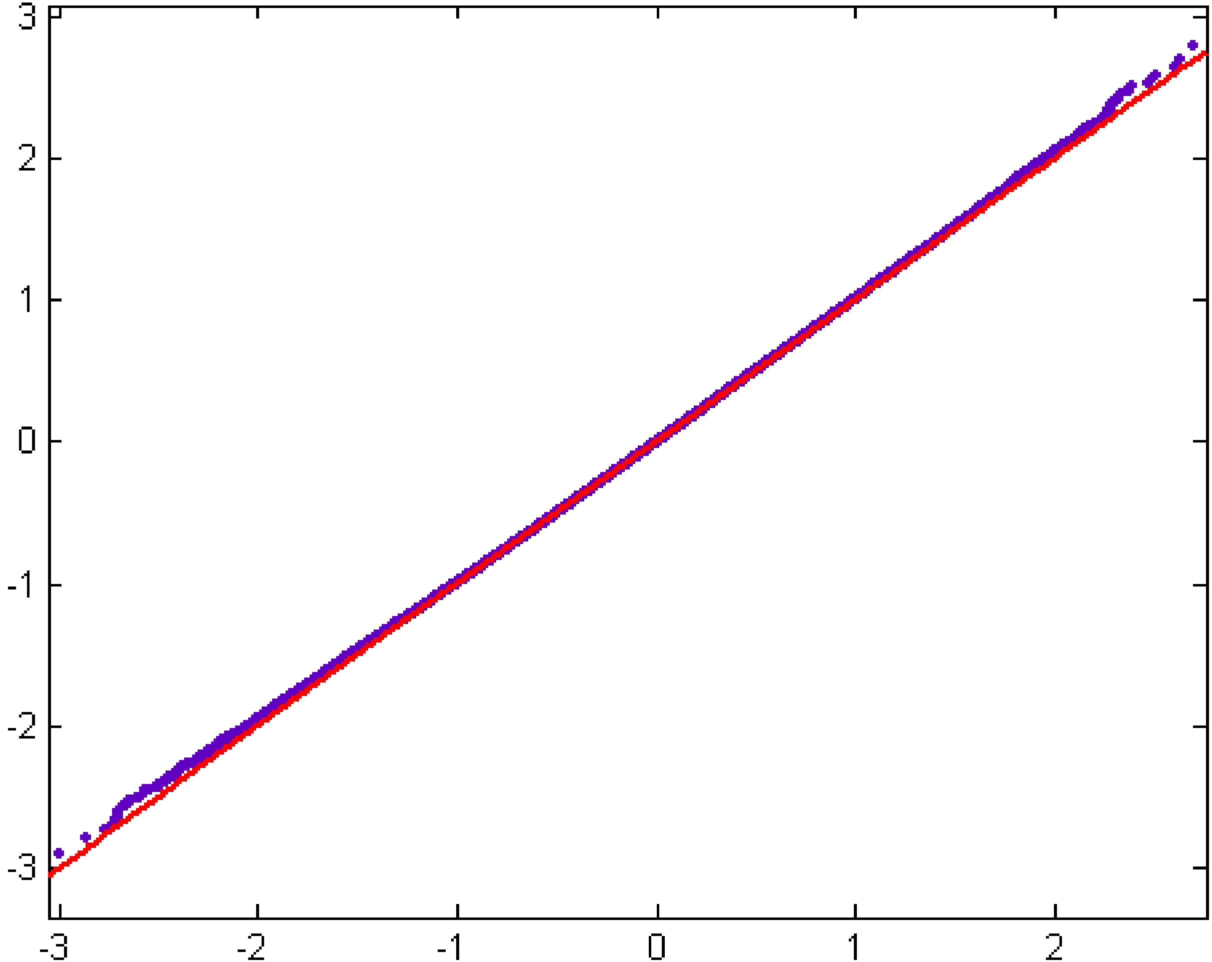


Figure 17 (n=p2qr, mean=0.0886, standard deviation=1.2953, size=87337)
6
4
2
0
-2
-4
-6
-5
-4
-3
-2
-1
0
1
2
3
4
5

Figure 18 (n=pq, mean=-0.0540, standard deviation=0.6444, size=197194)



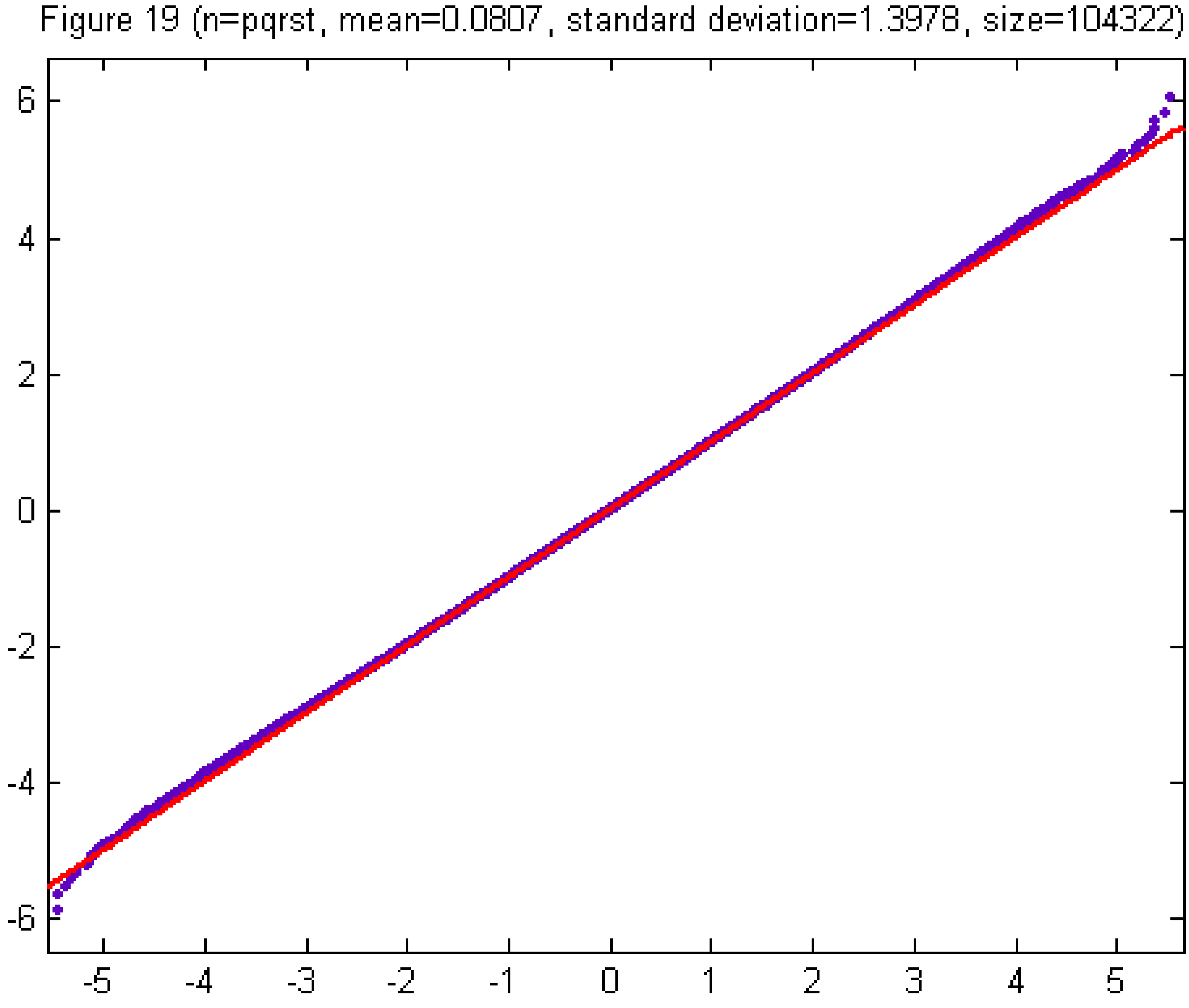


Figure 19 (n=pqrst, mean=0.0807, standard deviation=1.3978, size=104322)
6
4
2
0
-2
-4
-6
-5
-4
-3
-2
-1
0
1
2
3
4
5

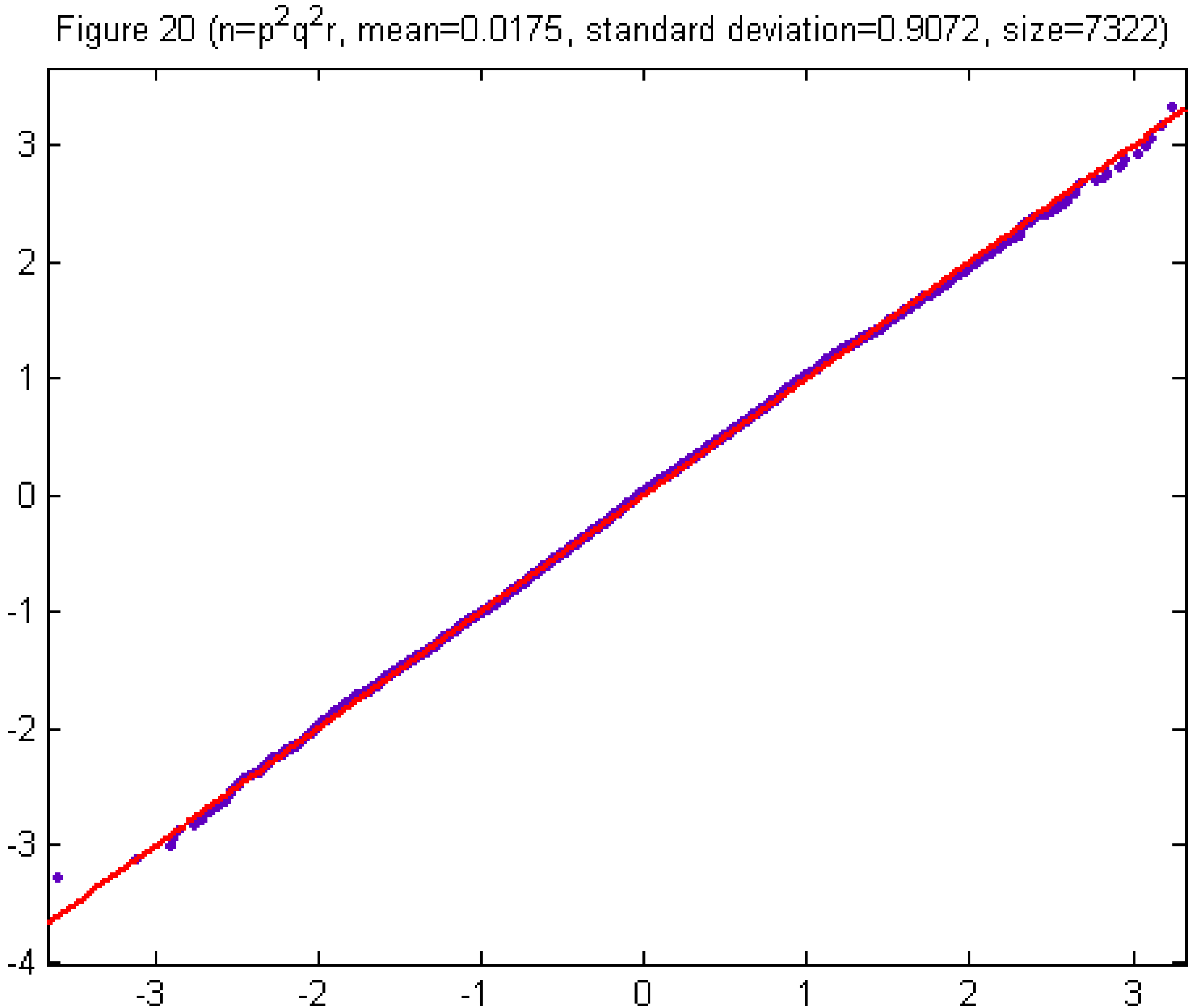
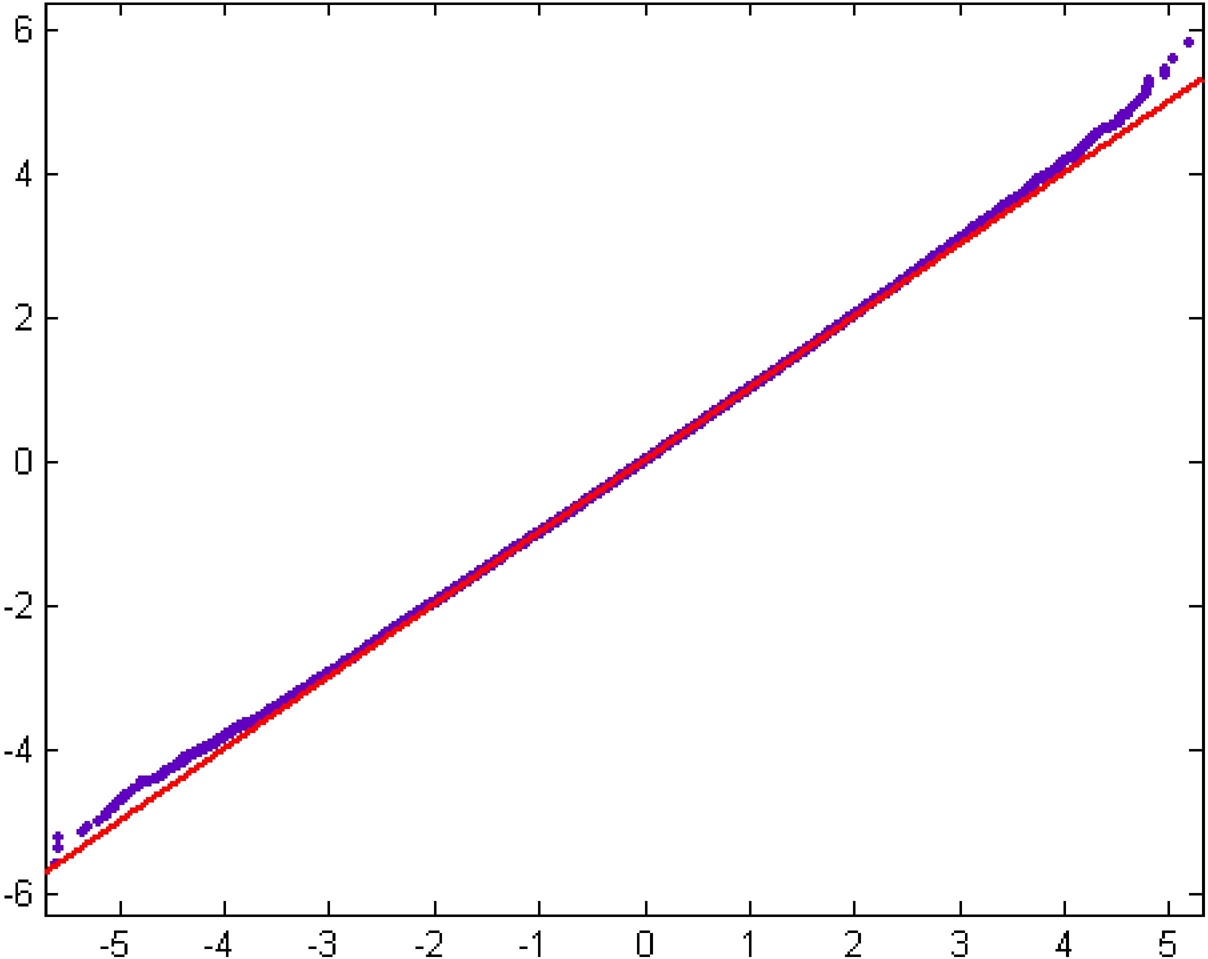


Figure 20 (n=p2q2r, mean=0.0175, standard deviation=0.9072, size=7322)
3
2
1
0
-1
-2
-3
-4
-3
-2
-1
0
1
2
3

Figure 21 ($n=p^2qrs$, mean=0.0977, standard deviation=1.3717, size=60656)



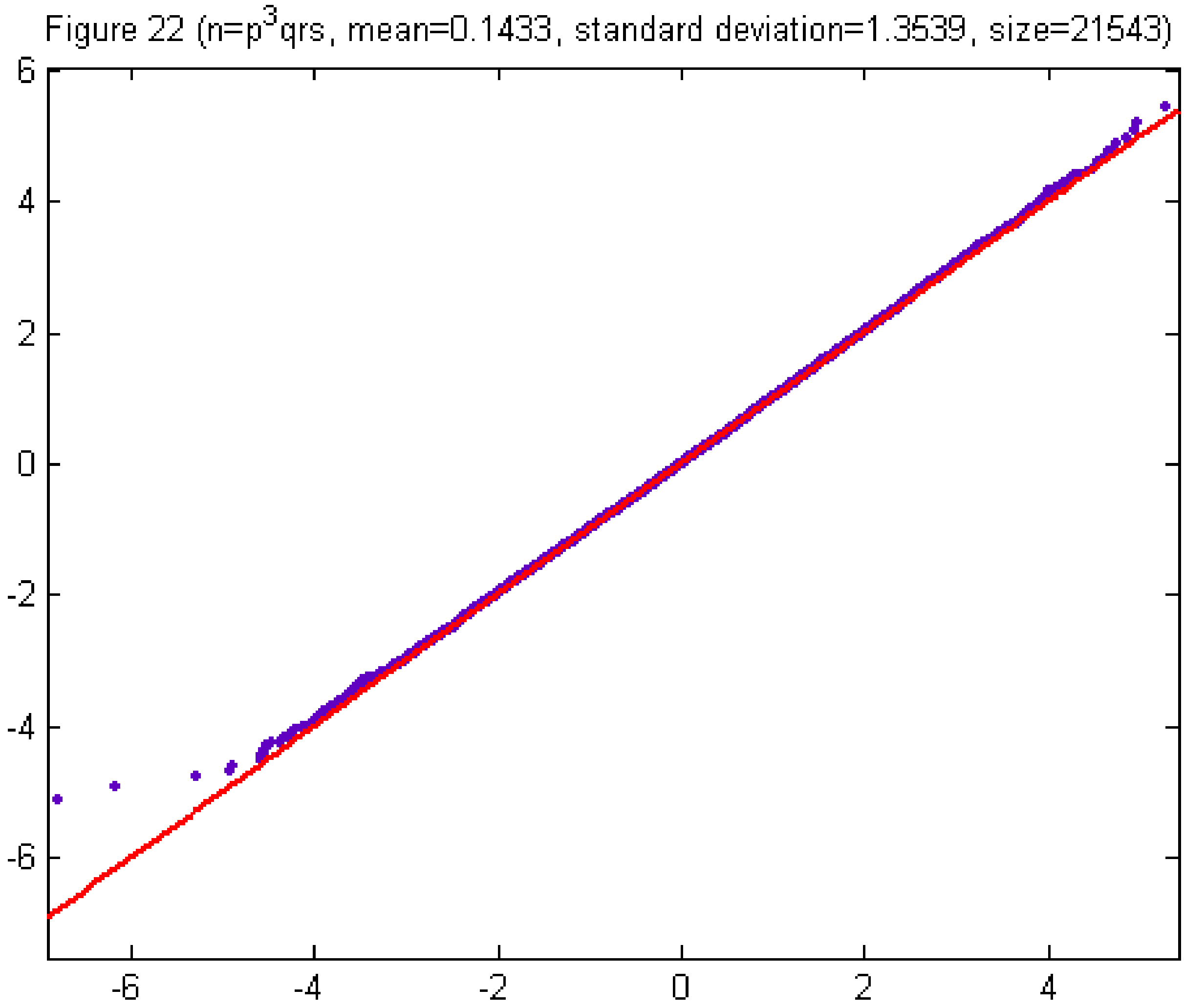
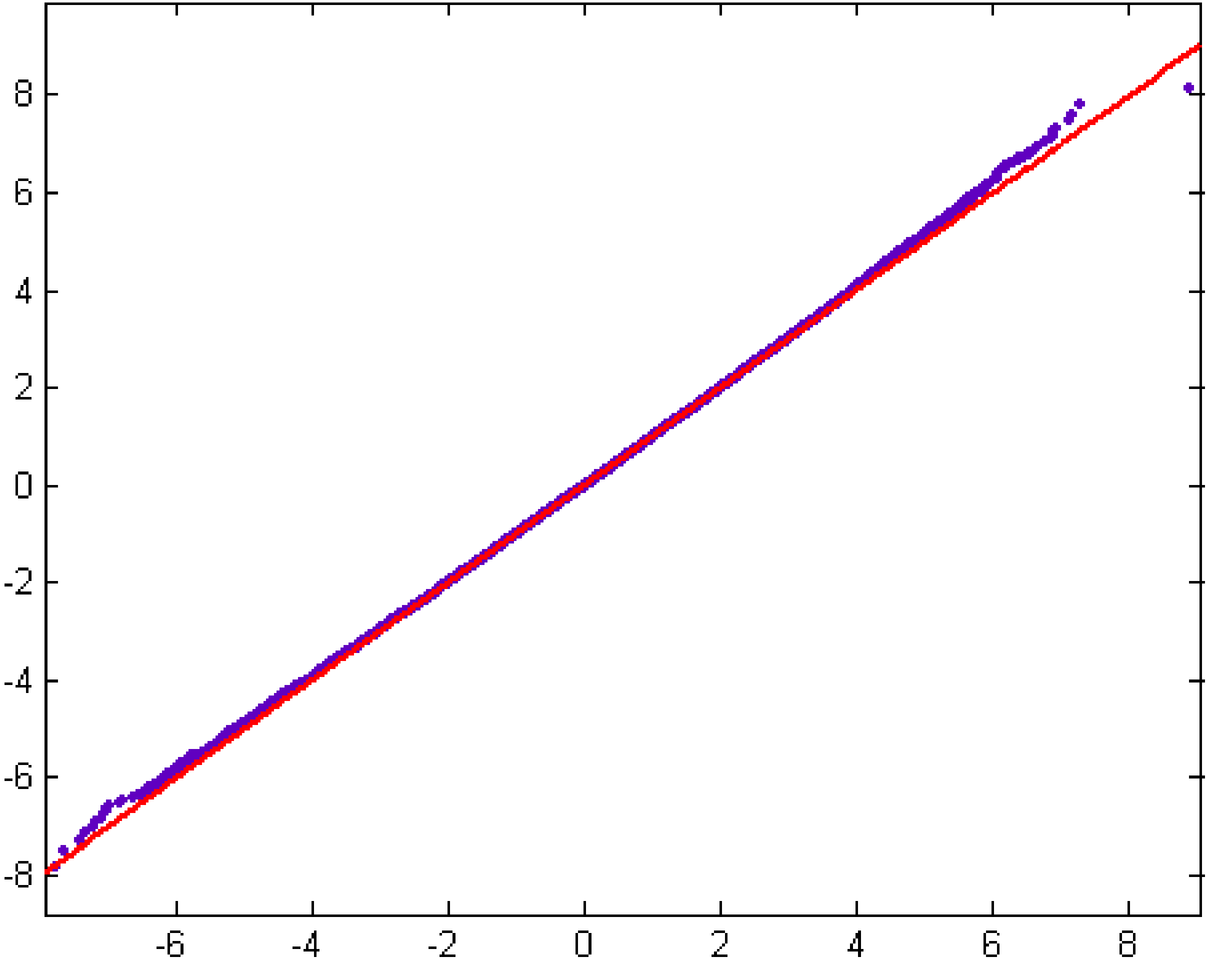


Figure 22 (n=p^3qrs, mean=0.1433, standard deviation=1.3539, size=21543)
6
4
2
0
-2
-4
-6
-6
-4
-2
0
2
4

Figure 23 (n=pqrst, mean=0.1548, standard deviation=2.0295, size=24568)



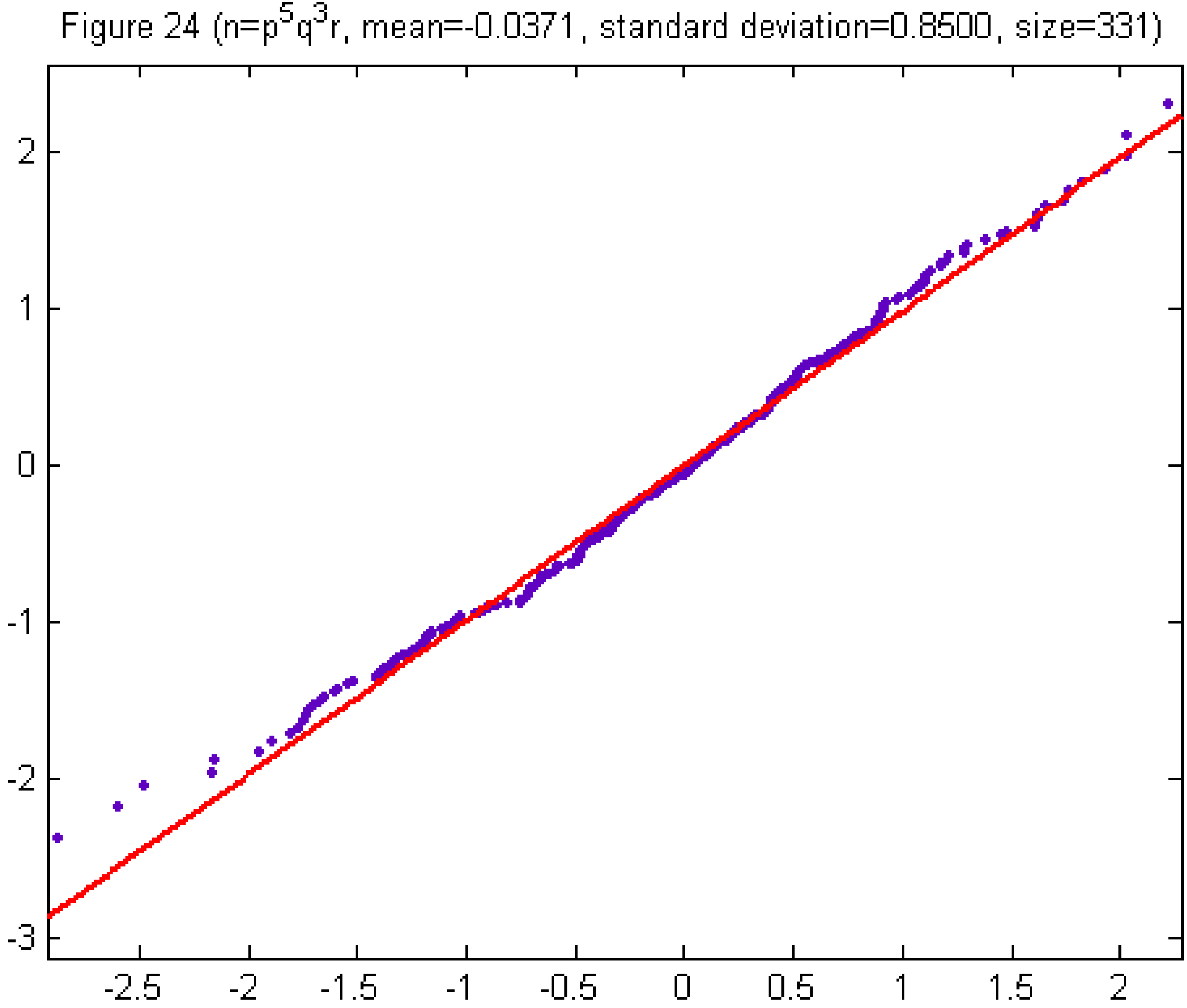


Figure 24 (n=p5q3r, mean=-0.0371, standard deviation=0.8500, size=331)
2
1
0
-1
-2
-3
-2.5
-2
-1.5
-1
-0.5
0
0.5
1
1.5
2

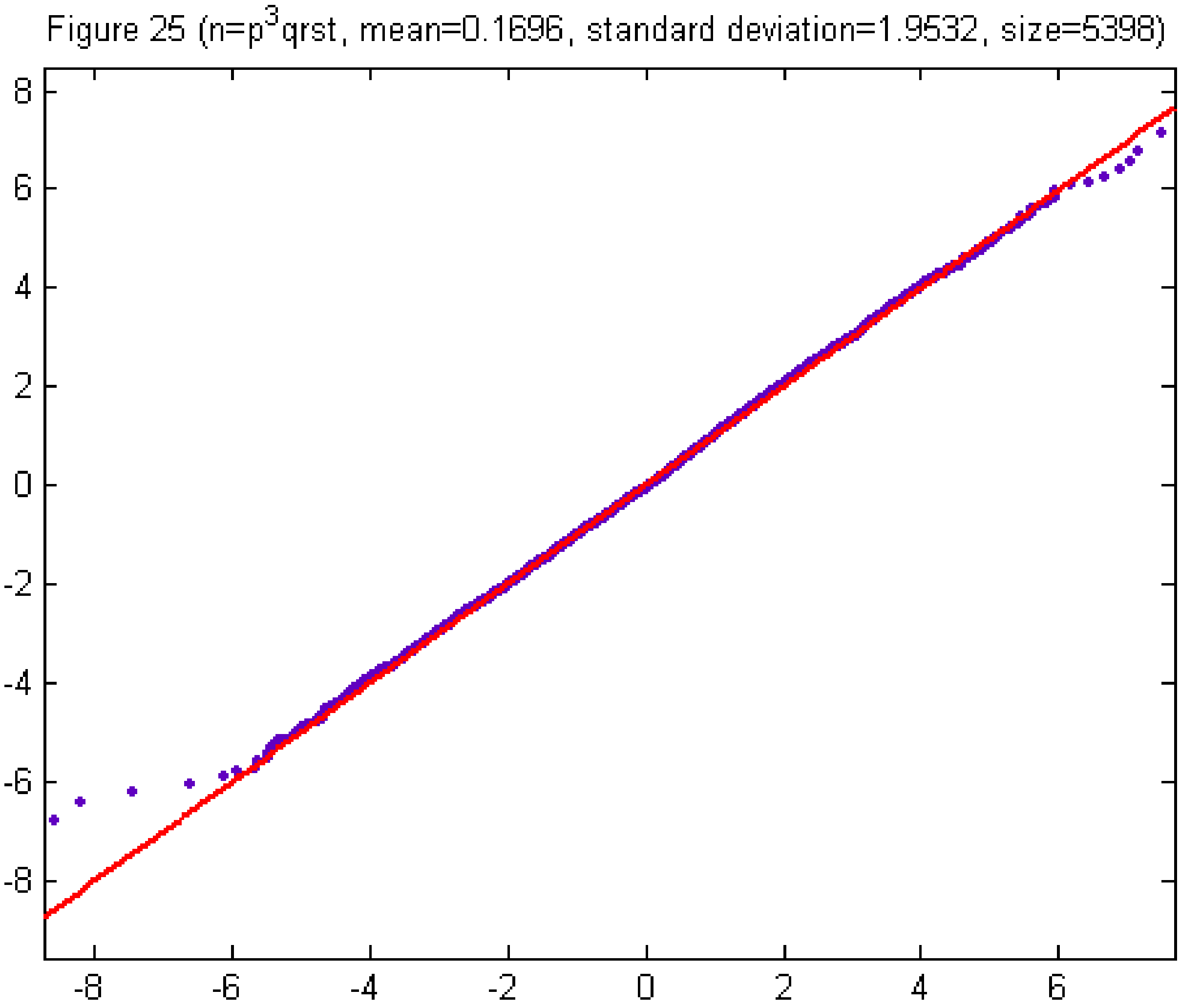


Figure 25 (n=p³qrst, mean=0.1696, standard deviation=1.9532, size=5398)
8
6
4
2
0
-2
-4
-6
-8
-8
-6
-4
-2
0
2
4
6

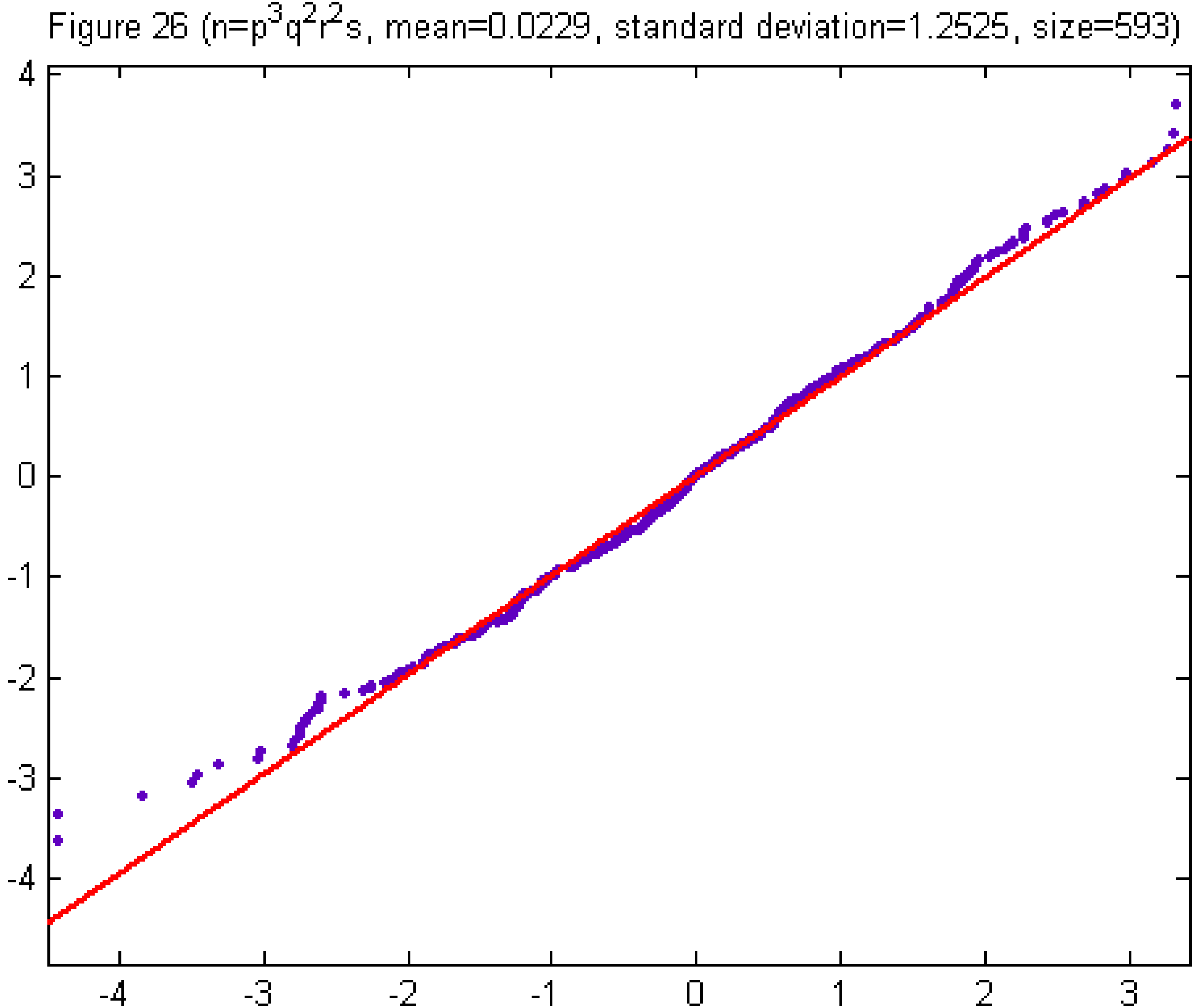
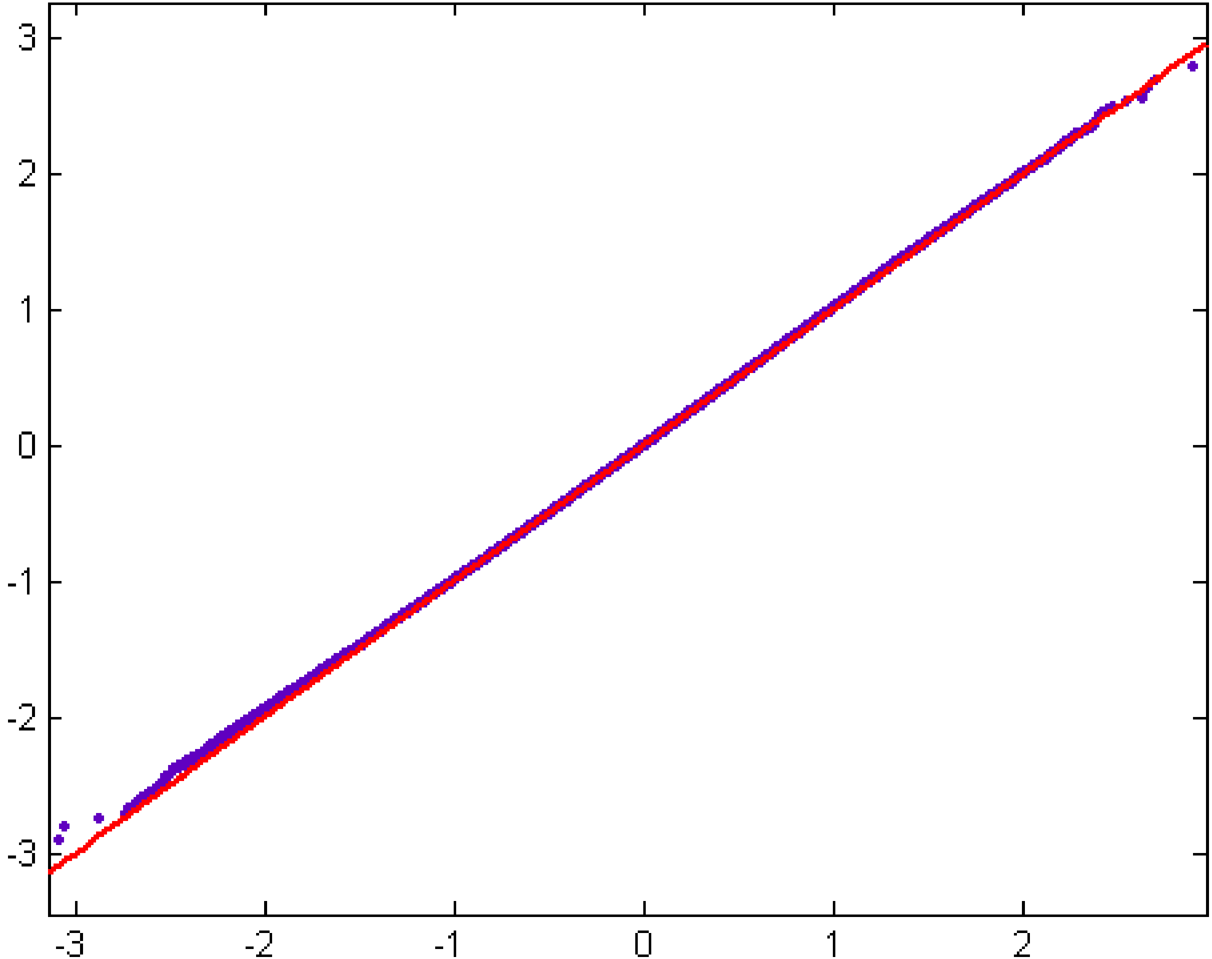


Figure 26 (n=p^3q^2r^2s, mean=0.0229, standard deviation=1.2525, size=593)
4
3
2
1
0
-1
-2
-3
-4
-4
-3
-2
-1
0
1
2
3

Figure 27 (n=pq, mean=-0.0544, standard deviation=0.6458, size=193116)



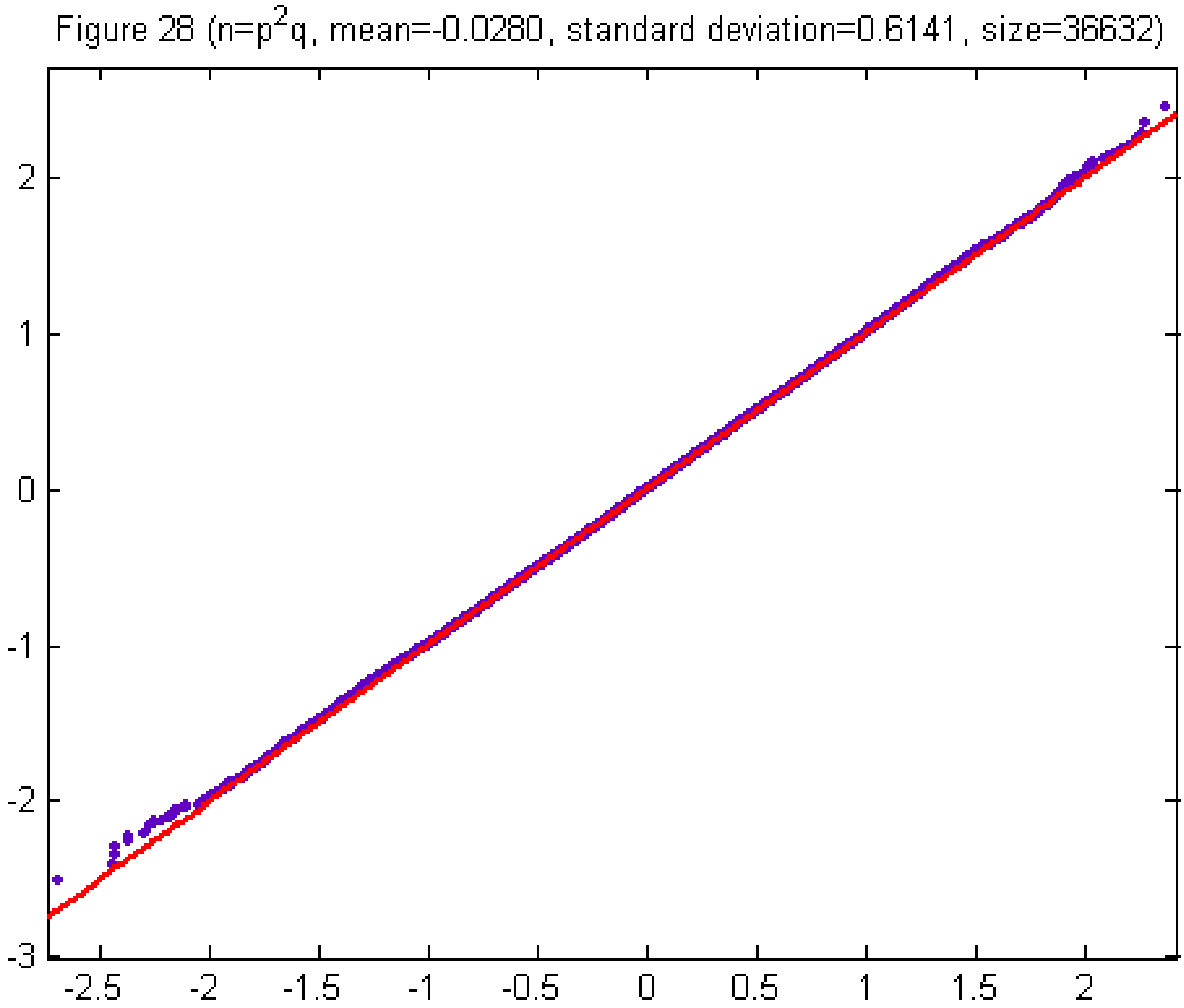
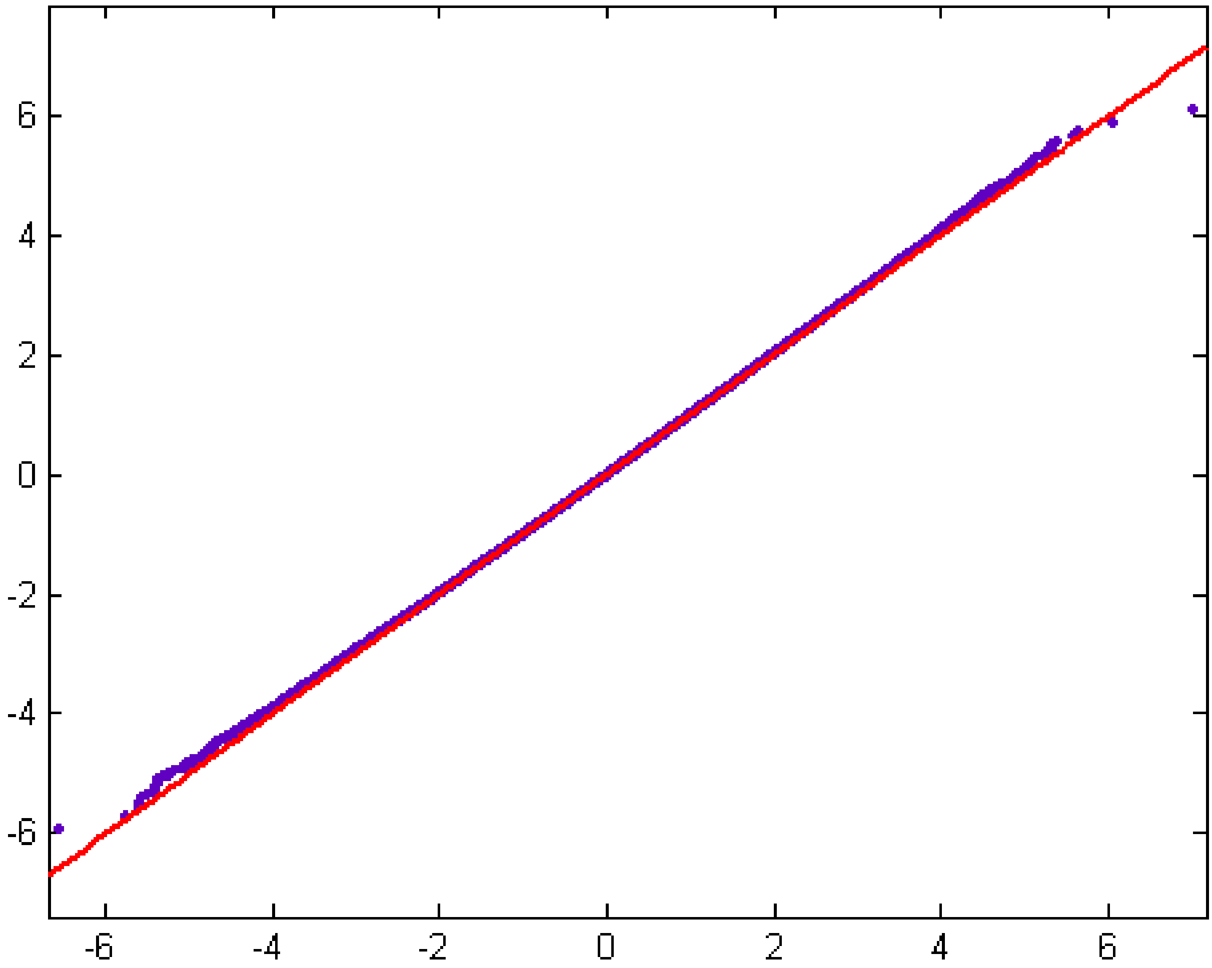


Figure 28 (n=p2q, mean=-0.0280, standard deviation=0.6141, size=36632)
2
1
0
-1
-2
-3
-2.5
-2
-1.5
-1
-0.5
0
0.5
1
1.5
2

Figure 29 (n=pqrs, mean=0.0802, standard deviation=1.4069, size=108253)



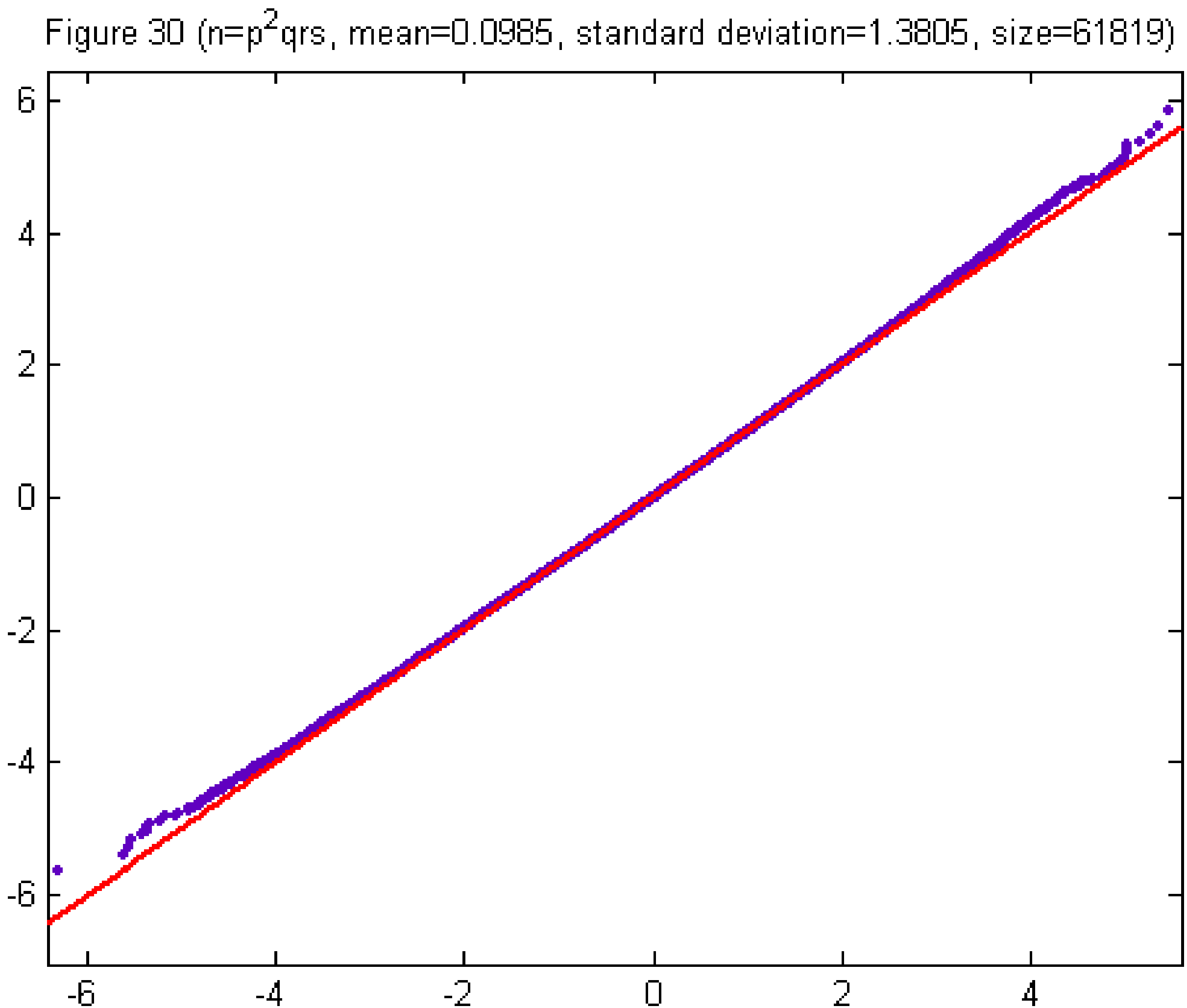
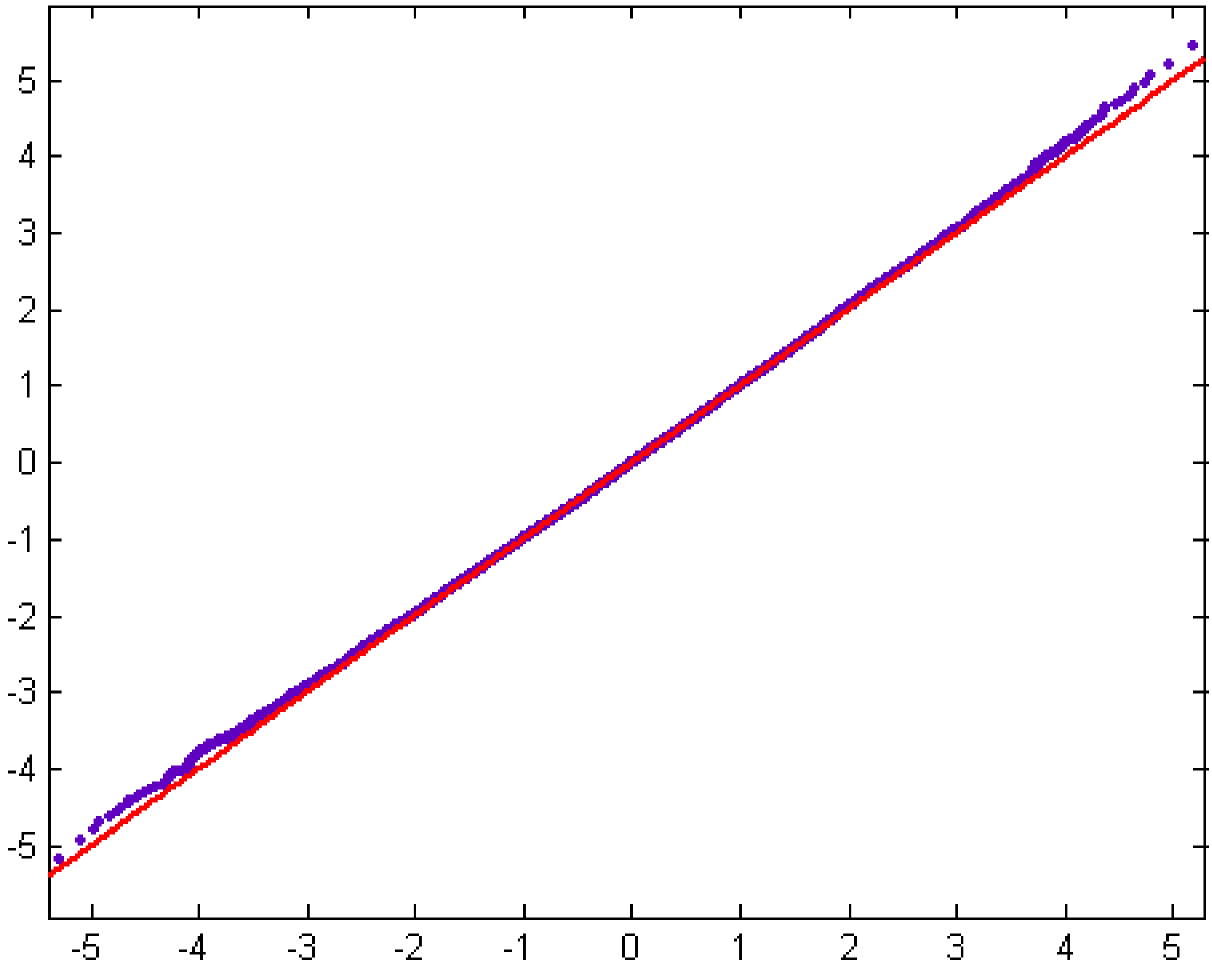


Figure 30 (n=p2qrs, mean=0.0985, standard deviation=1.3805, size=61819)
6
4
2
0
-2
-4
-6
-6
-4
-2
0
2
4

Figure 31 ($n=p^3qrs$, mean=0.1259, standard deviation=1.3582, size=22100)



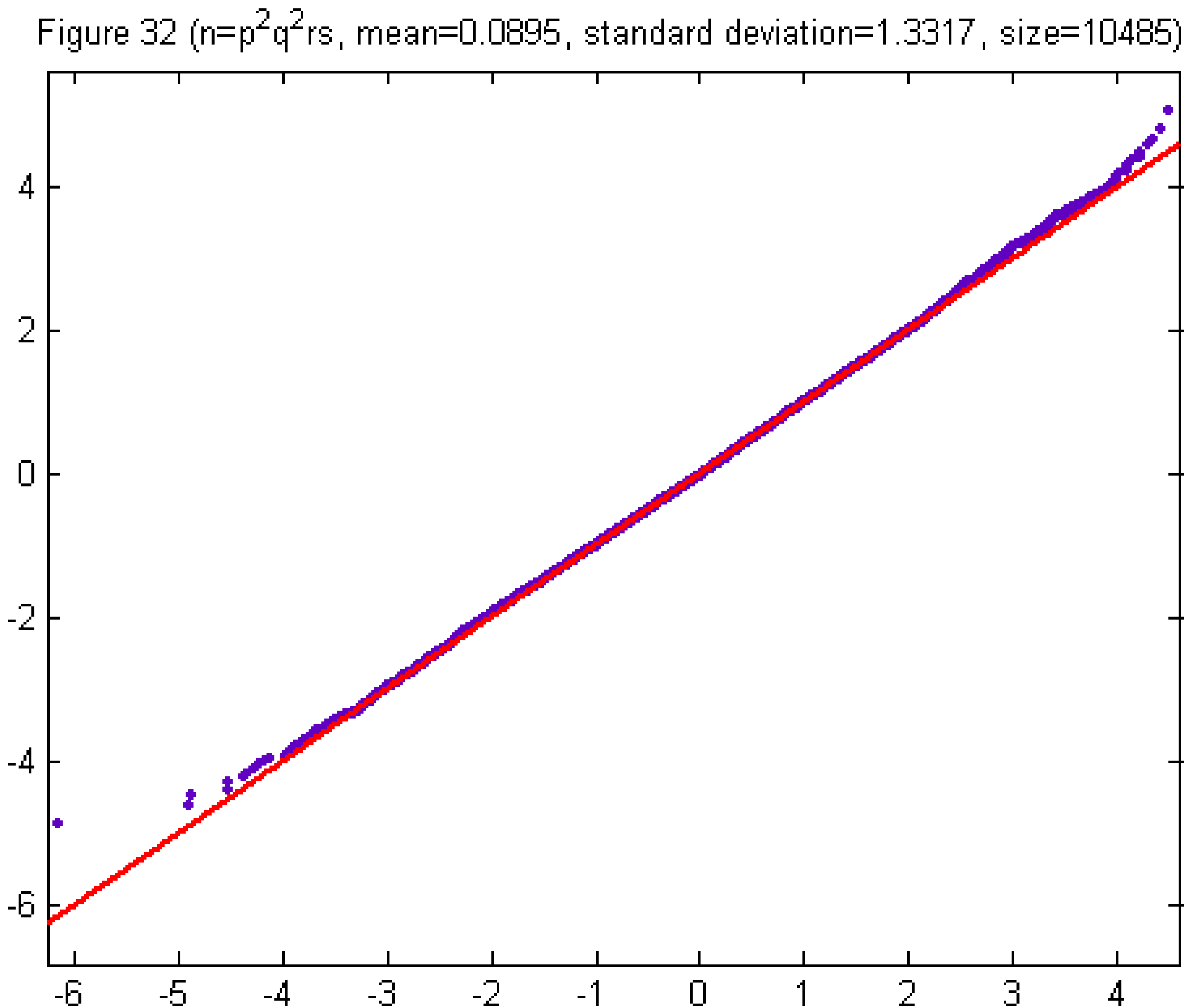
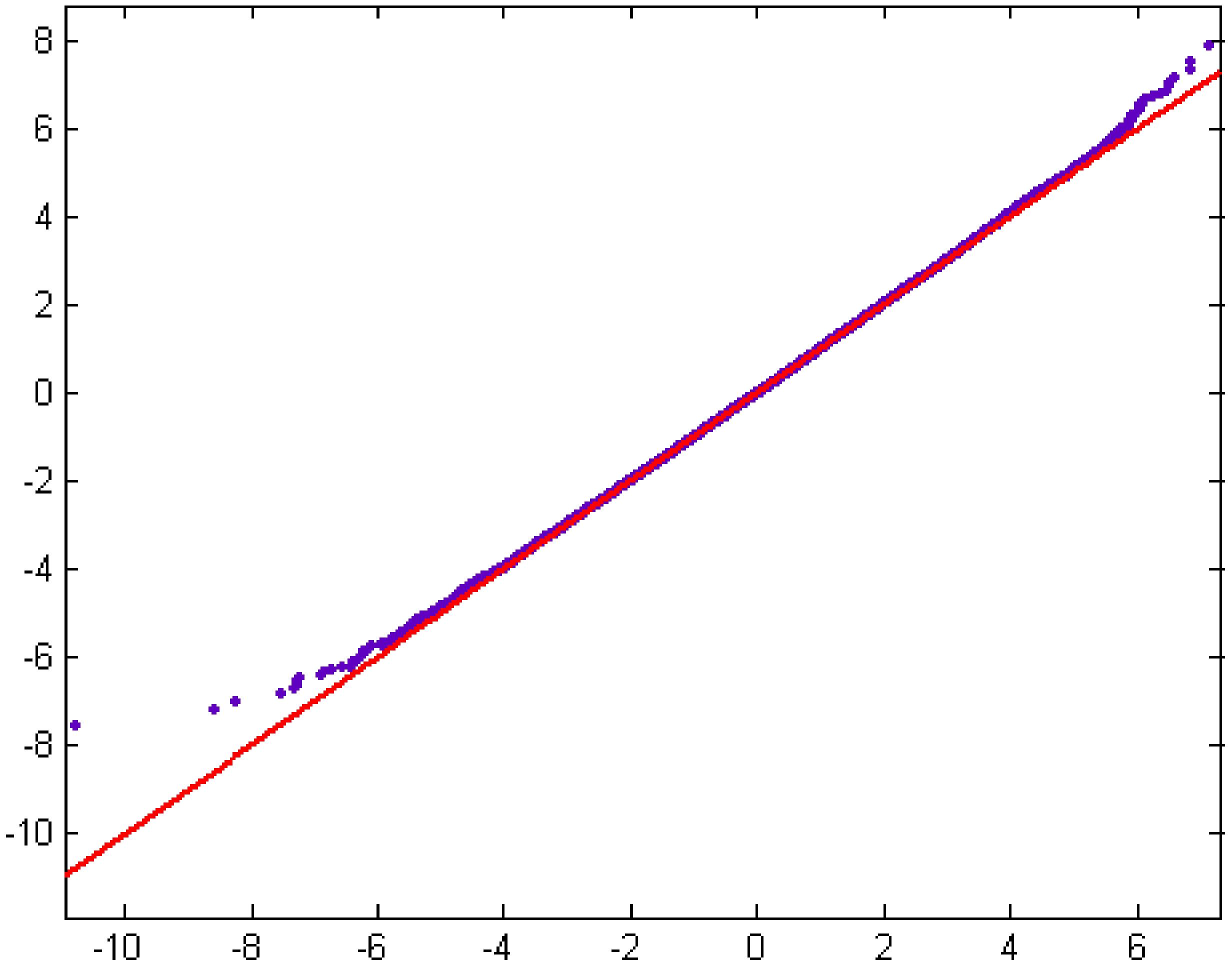


Figure 32 (n=p2q2rs, mean=0.0895, standard deviation=1.3317, size=10485)
4
2
0
-2
-4
-6
-6
-5
-4
-3
-2
-1
0
1
2
3
4

Figure 33 ($n=p^2qrst$, mean=0.1613, standard deviation=1.9899, size=18761)



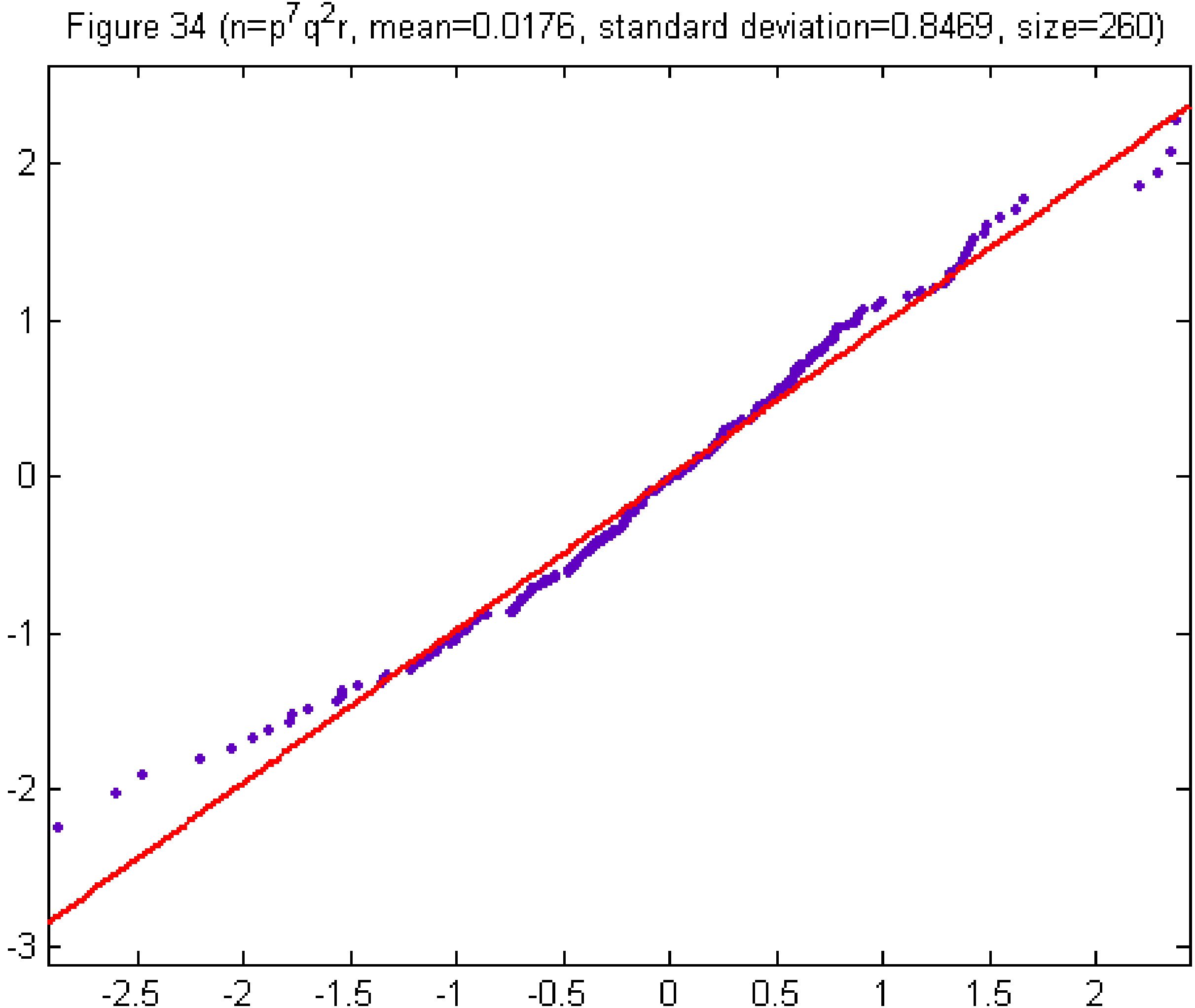
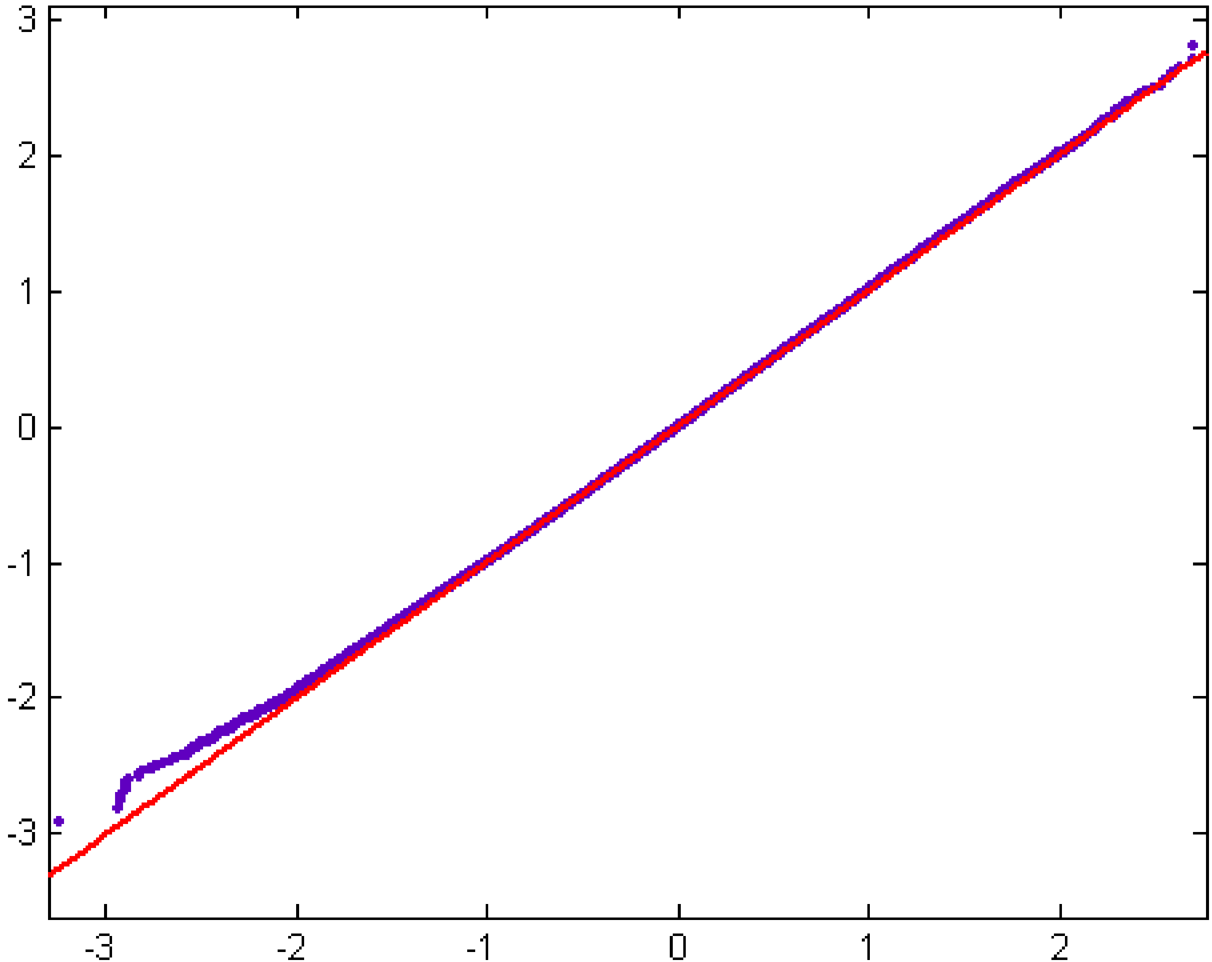


Figure 34 (n=p^7q^2r, mean=0.0176, standard deviation=0.8469, size=260)
2
1
0
-1
-2
-3
-2.5
-2
-1.5
-1
-0.5
0
0.5
1
1.5
2

Figure 35 (n=pq, mean=-0.0546, standard deviation=0.6485, size=190506)



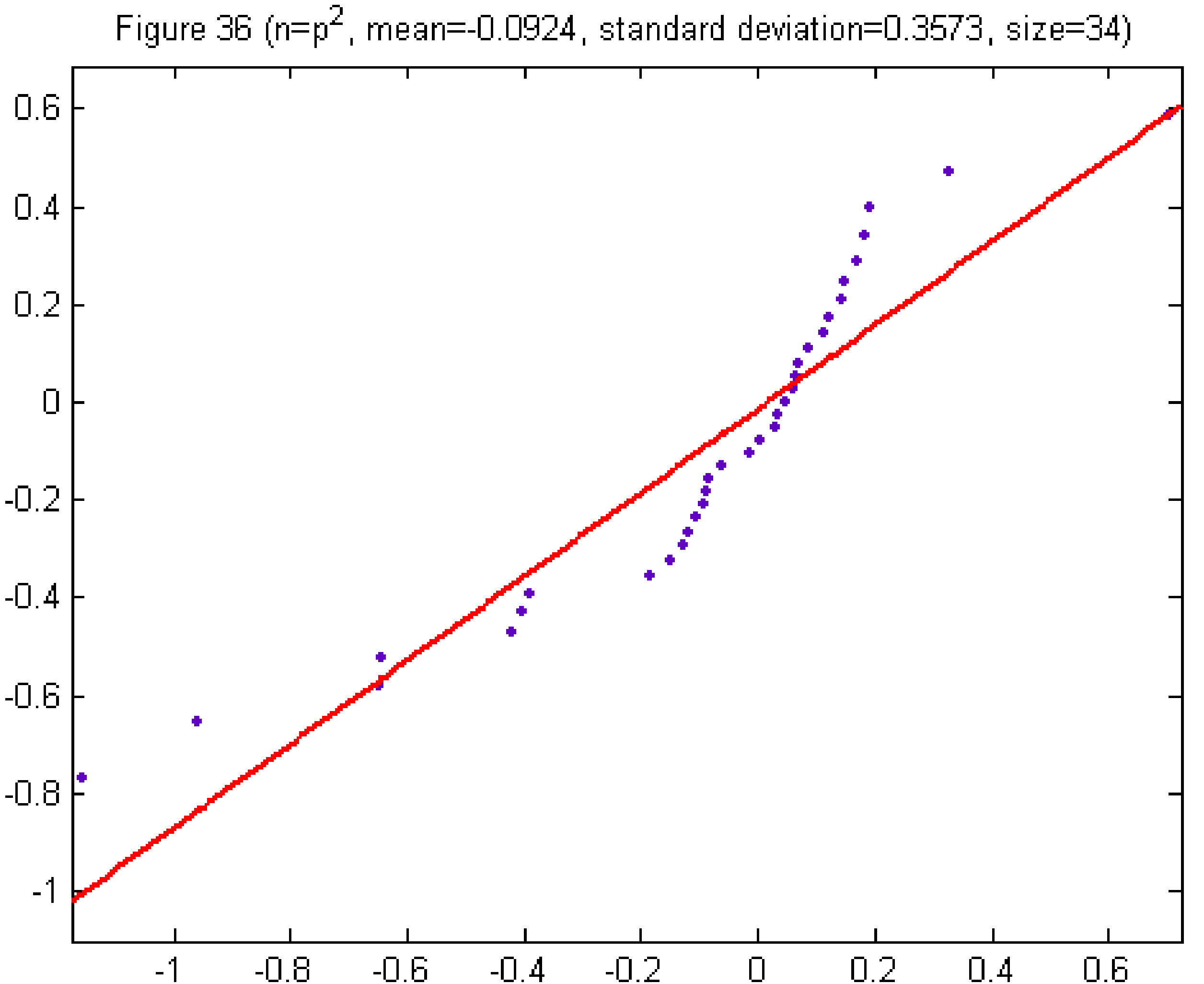


Figure 36 (n=p^2, mean=-0.0924, standard deviation=0.3573, size=34)
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
-1
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
0.6

Figure 37 ($n=p^3q$, mean=0.0083, standard deviation=0.6124, size=14110)

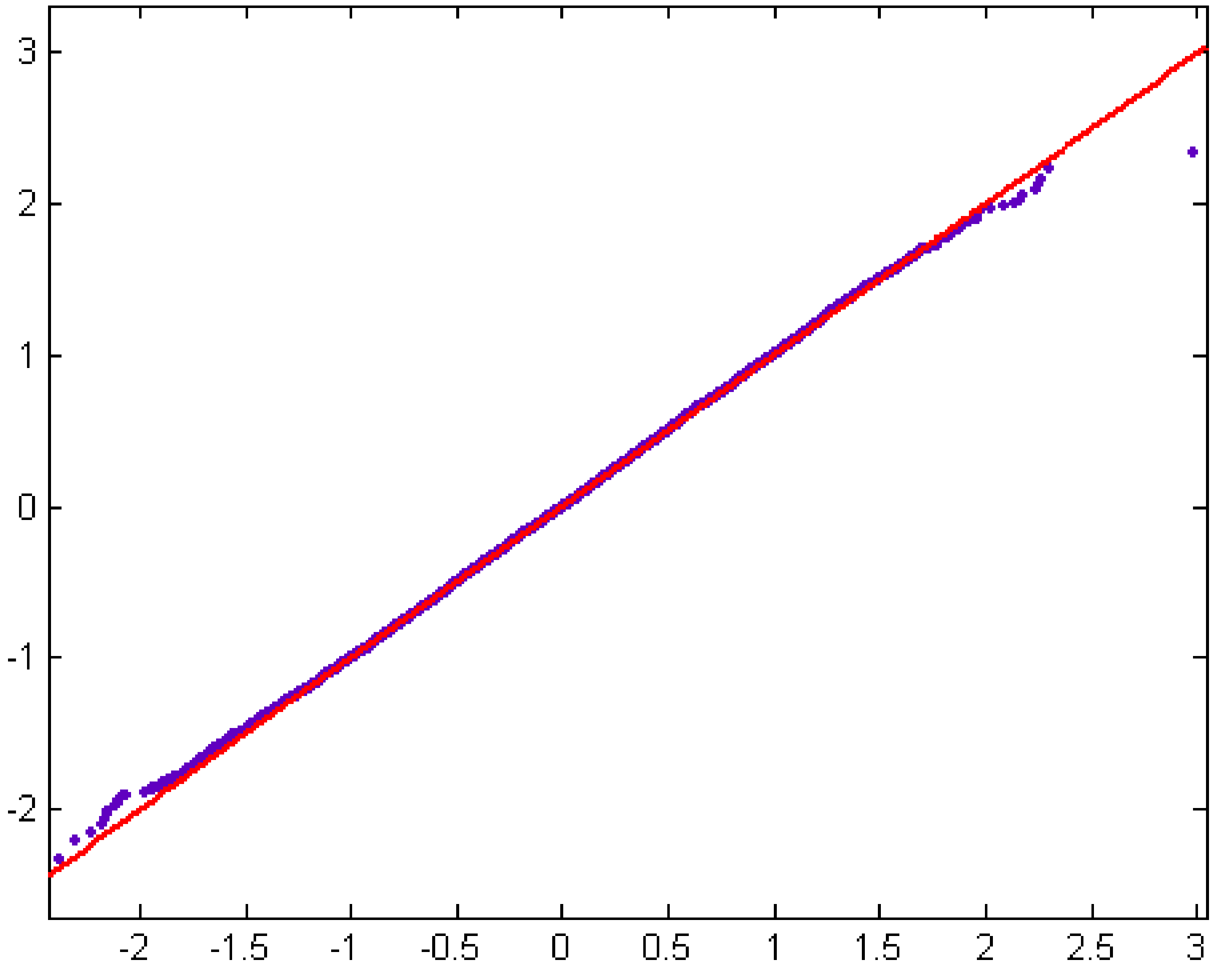
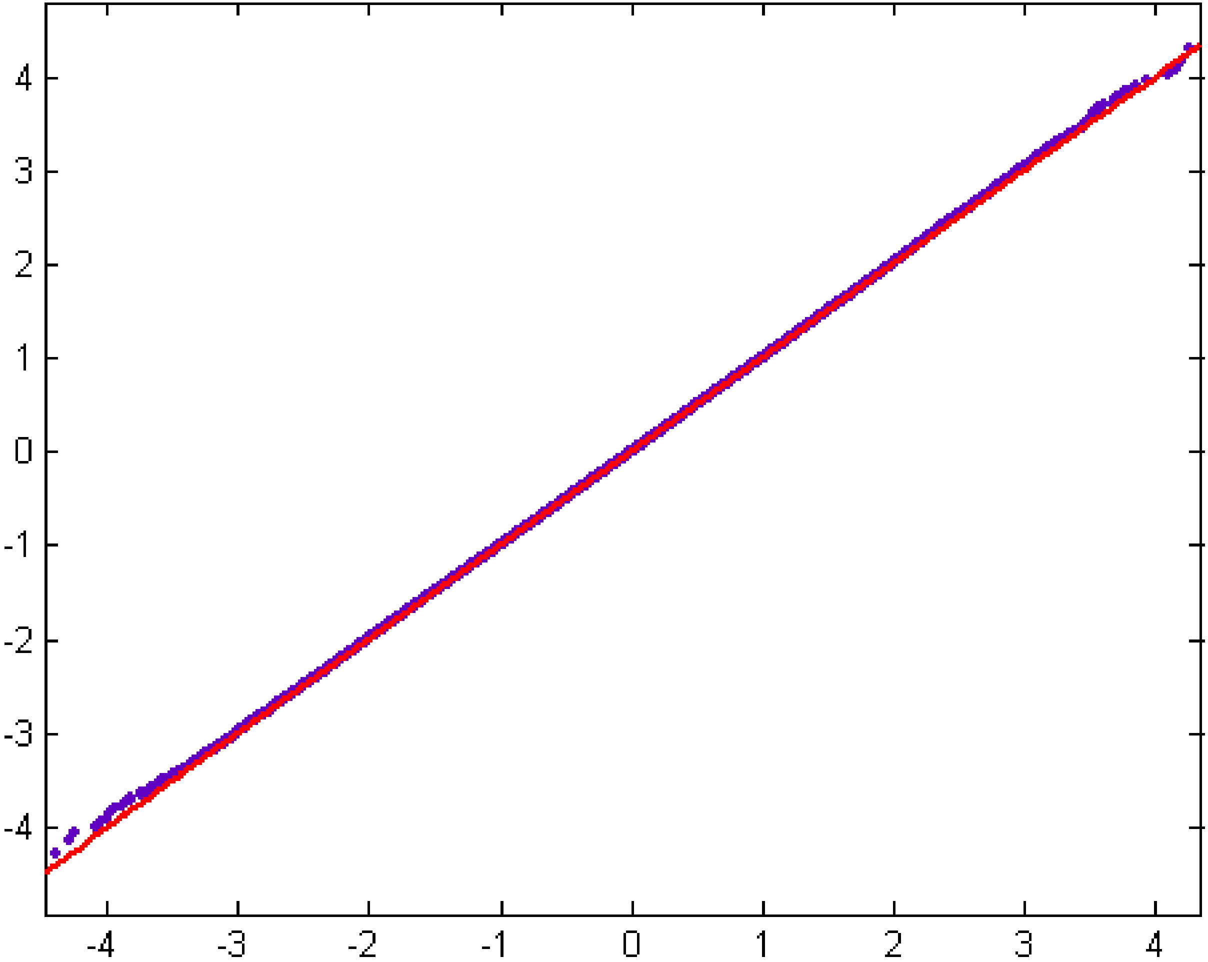


Figure 38 (n=pqr, mean=0.0067, standard deviation=0.9721, size=209072)



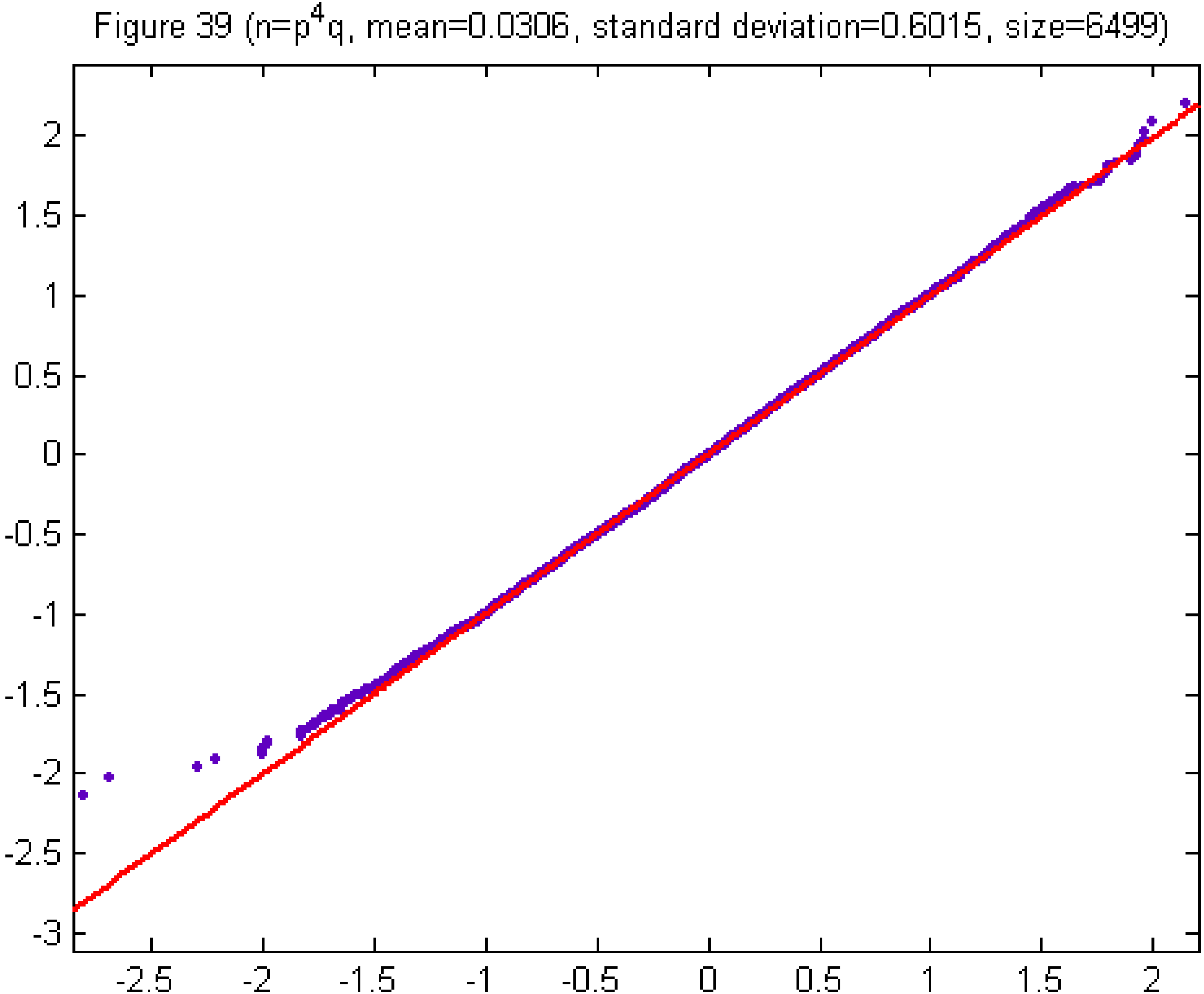


Figure 39 (n=p4q, mean=0.0306, standard deviation=0.6015, size=6499)
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
-2.5
-3
-2.5
-2
-1.5
-1
-0.5
0
0.5
1
1.5
2

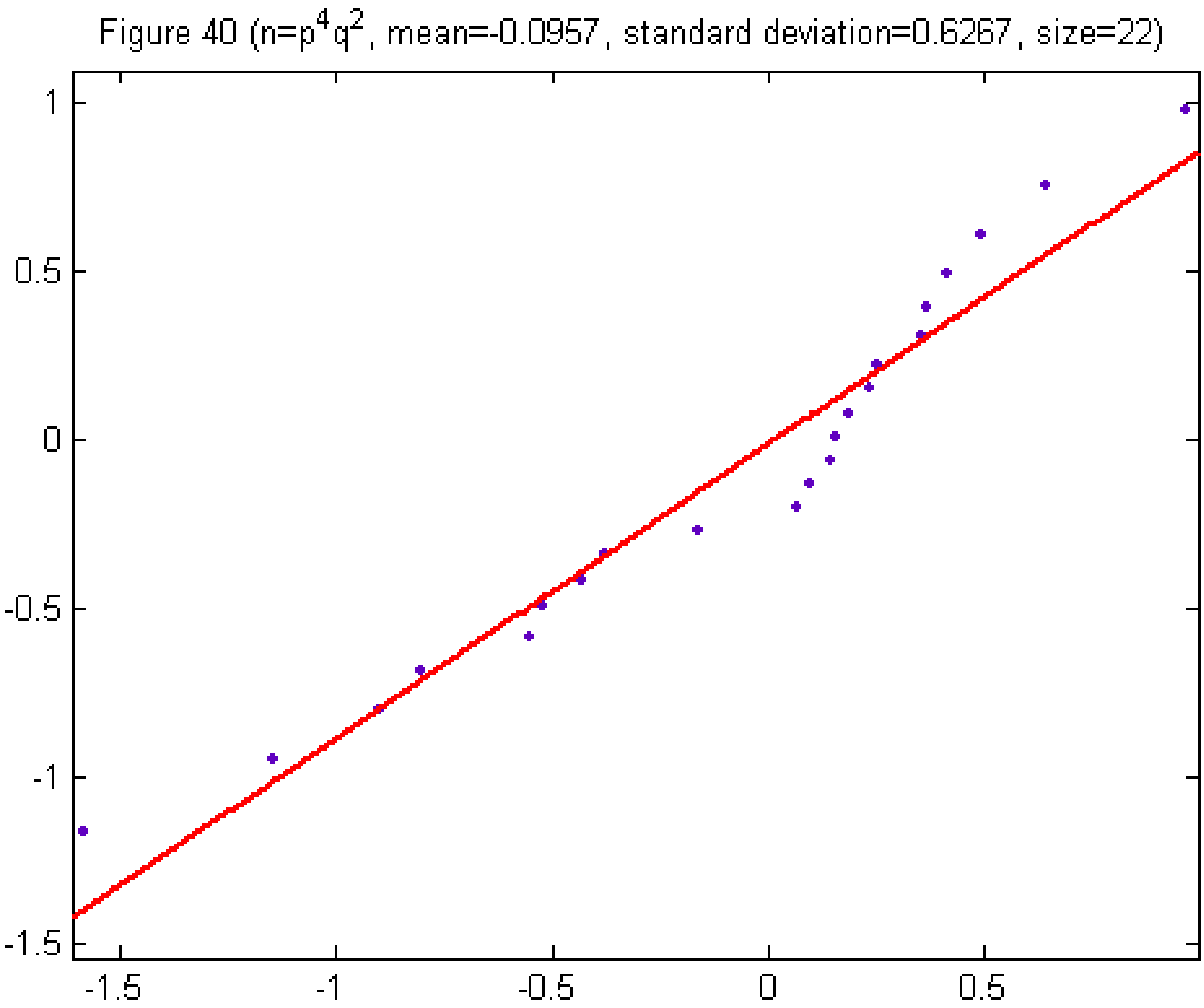


Figure 40 (n=p4q2, mean=-0.0957, standard deviation=0.6267, size=22)
1
0.5
0
-0.5
-1
-1.5
-1.5
-1
-0.5
0
0.5

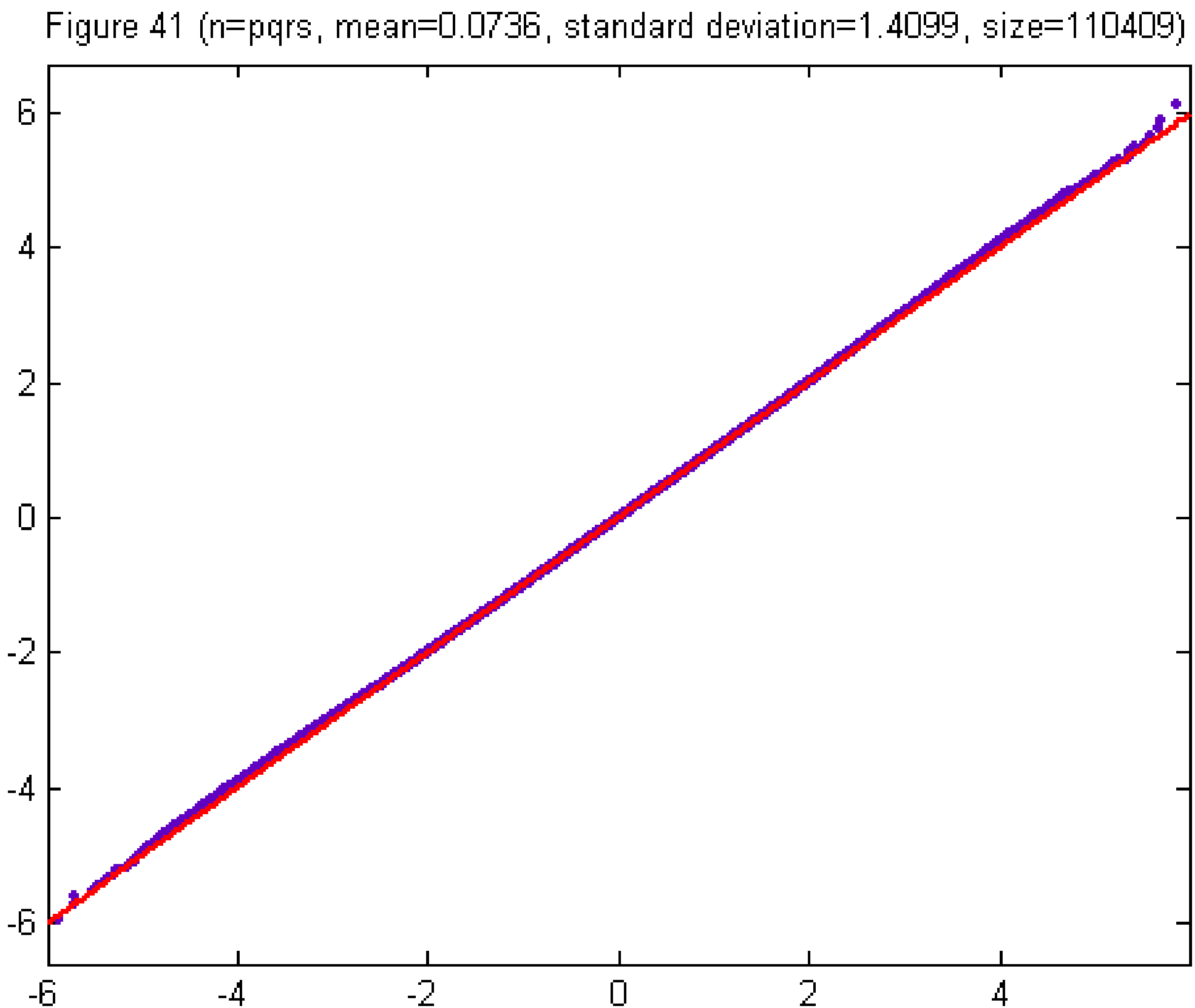


Figure 41 (n=pqrs, mean=0.0736, standard deviation=1.4099, size=110409)
6
4
2
0
-2
-4
-6
-6
-4
-2
0
2
4

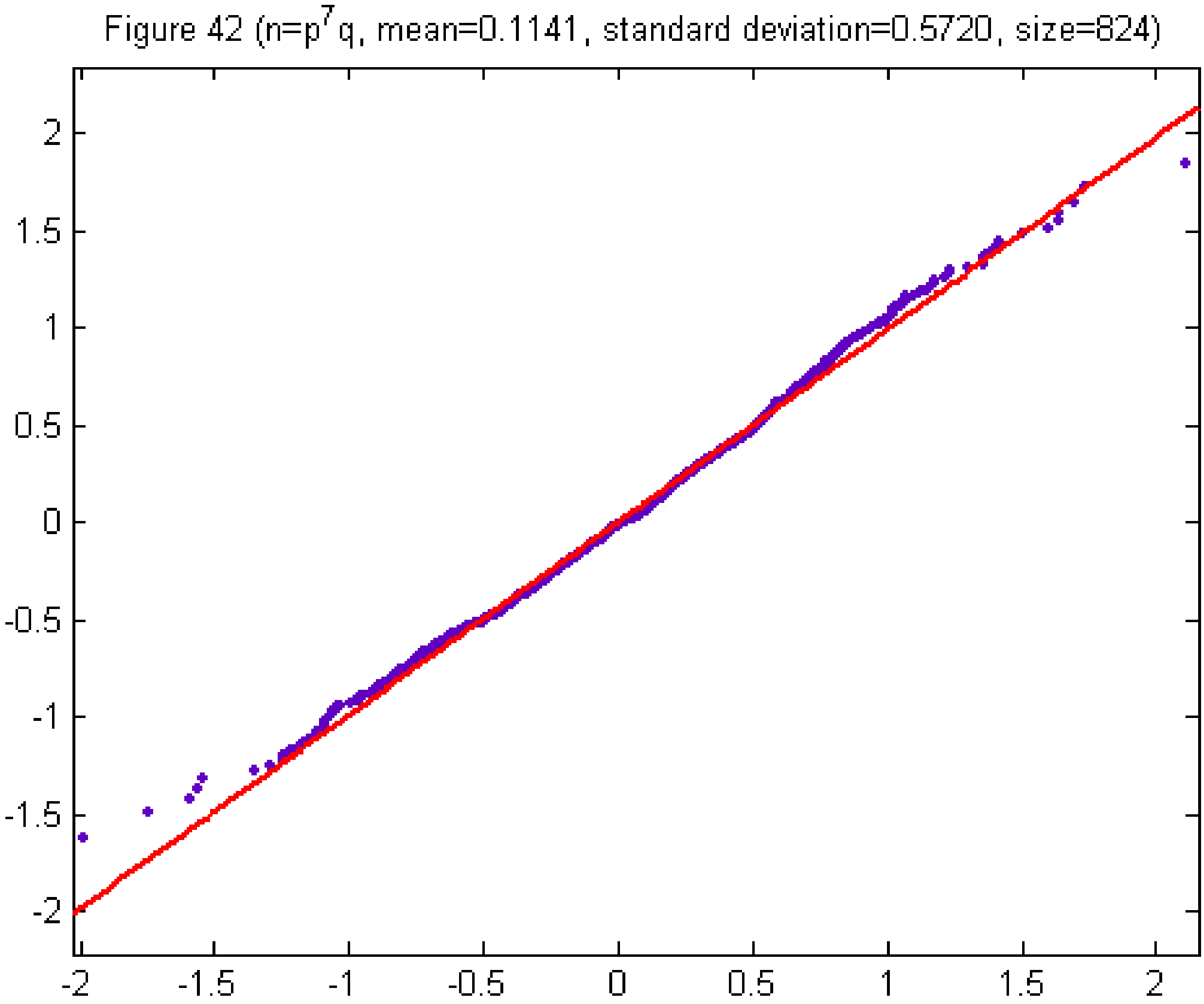
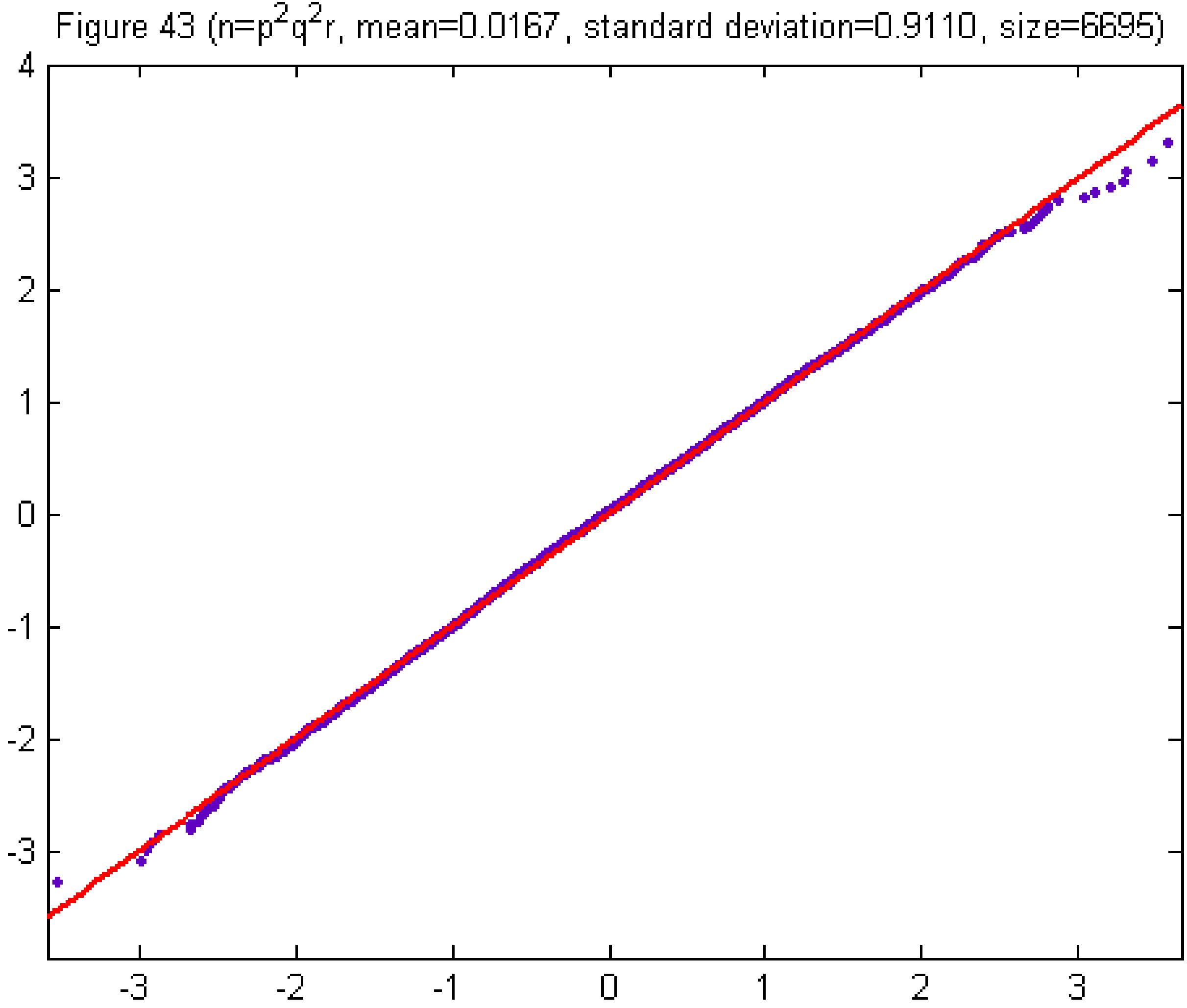


Figure 42 (n=p7q, mean=0.1141, standard deviation=0.5720, size=824)
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
-2
-1.5
-1
-0.5
0
0.5
1
1.5
2



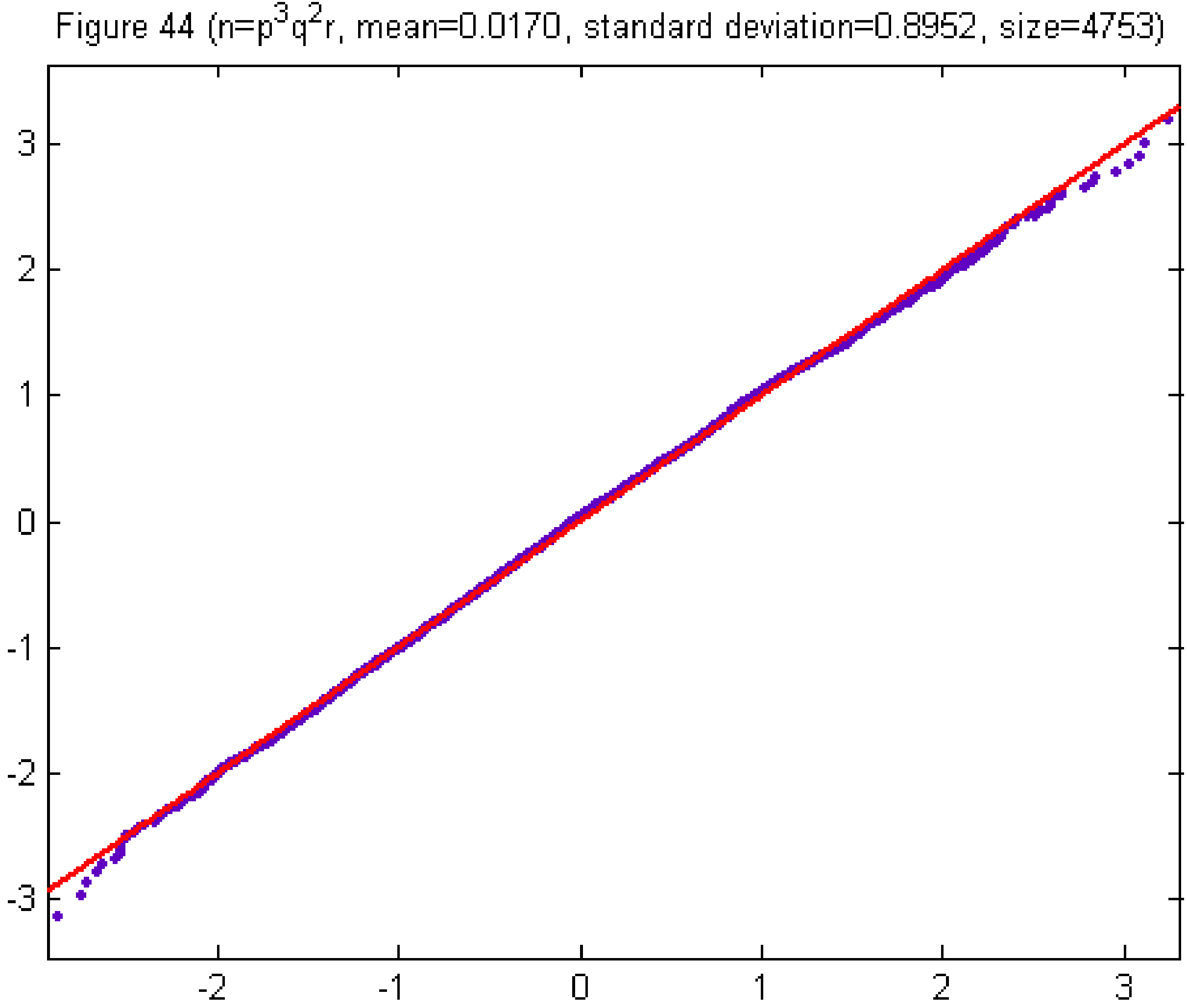
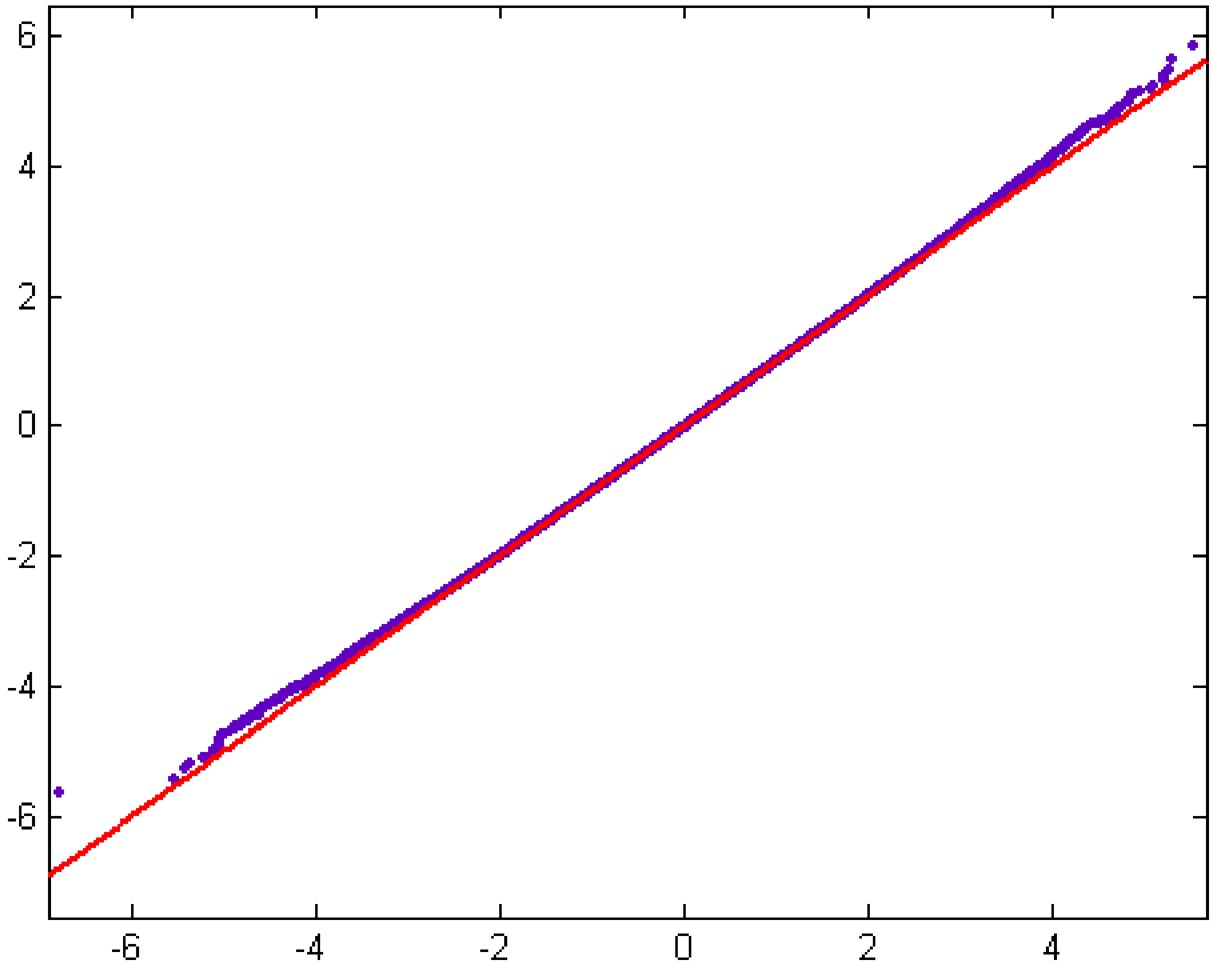


Figure 44 (n=p3q2r, mean=0.0170, standard deviation=0.8952, size=4753)
3
2
1
0
-1
-2
-3
-2
-1
0
1
2
3

Figure 45 ($n=p^2qrs$, mean=0.0991, standard deviation=1.3829, size=62558)



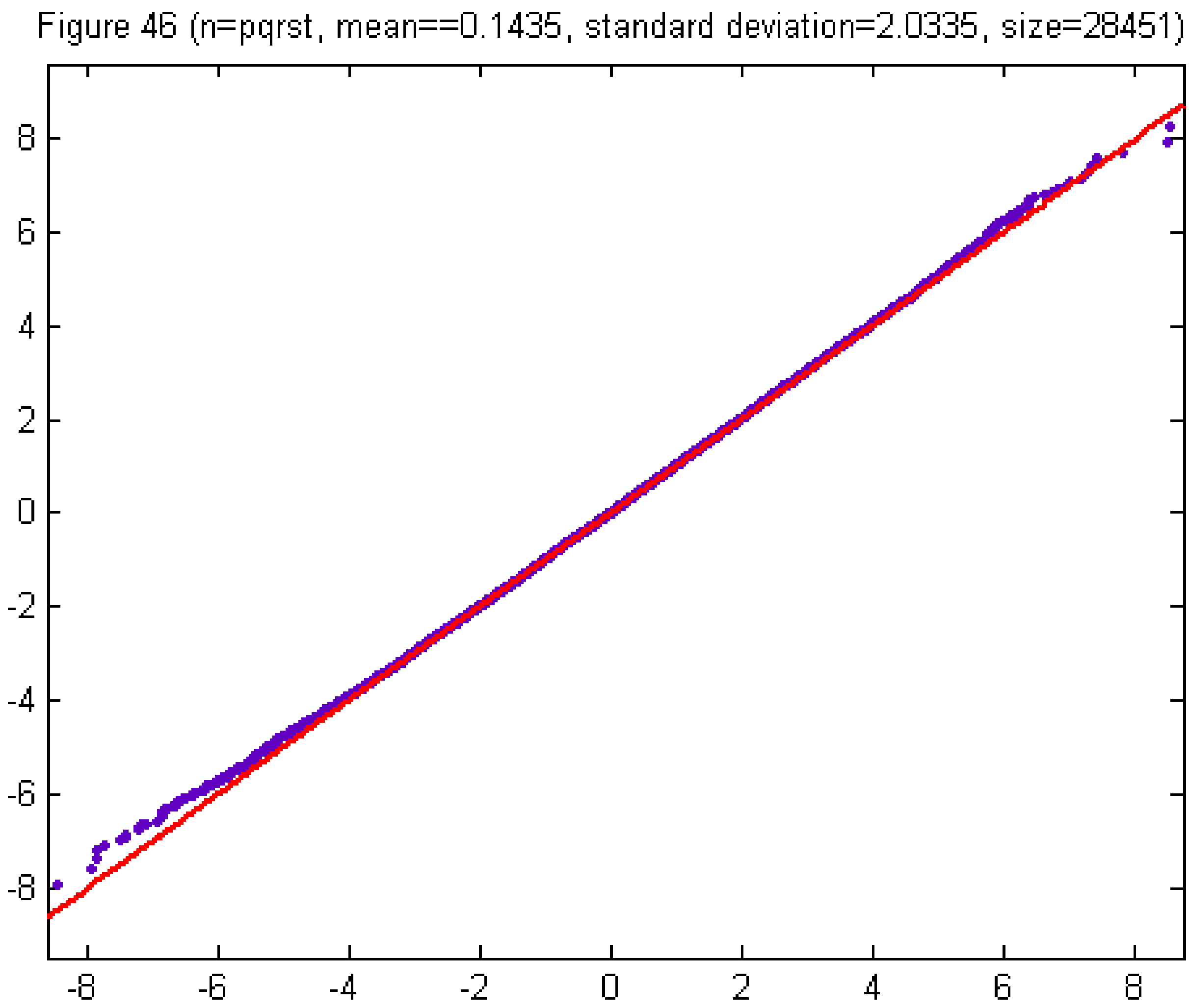


Figure 46 (n=pqrst, mean==0.1435, standard deviation=2.0335, size=28451)
8
6
4
2
0
-2
-4
-6
-8
-8
-6
-4
-2
0
2
4
6
8

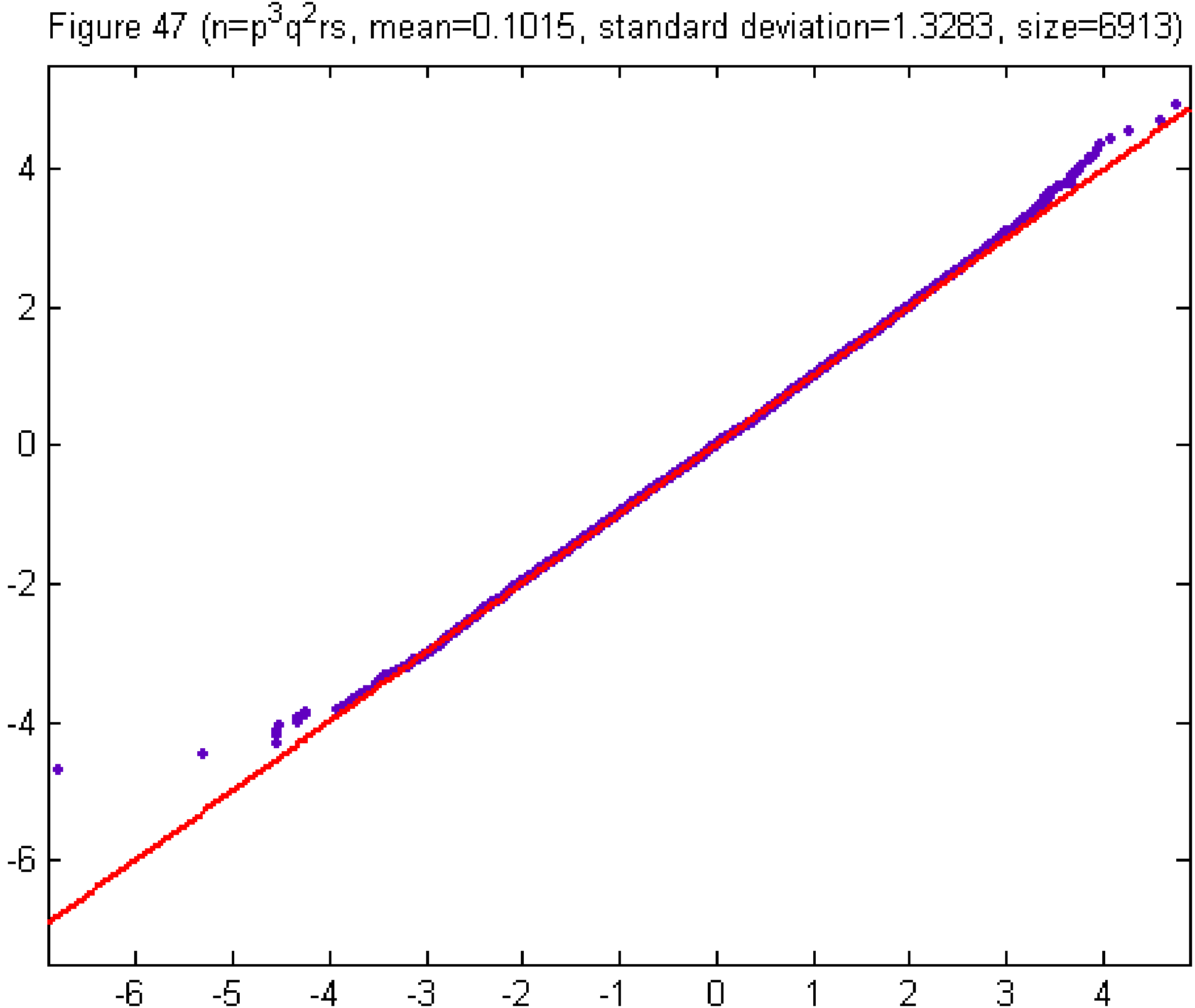
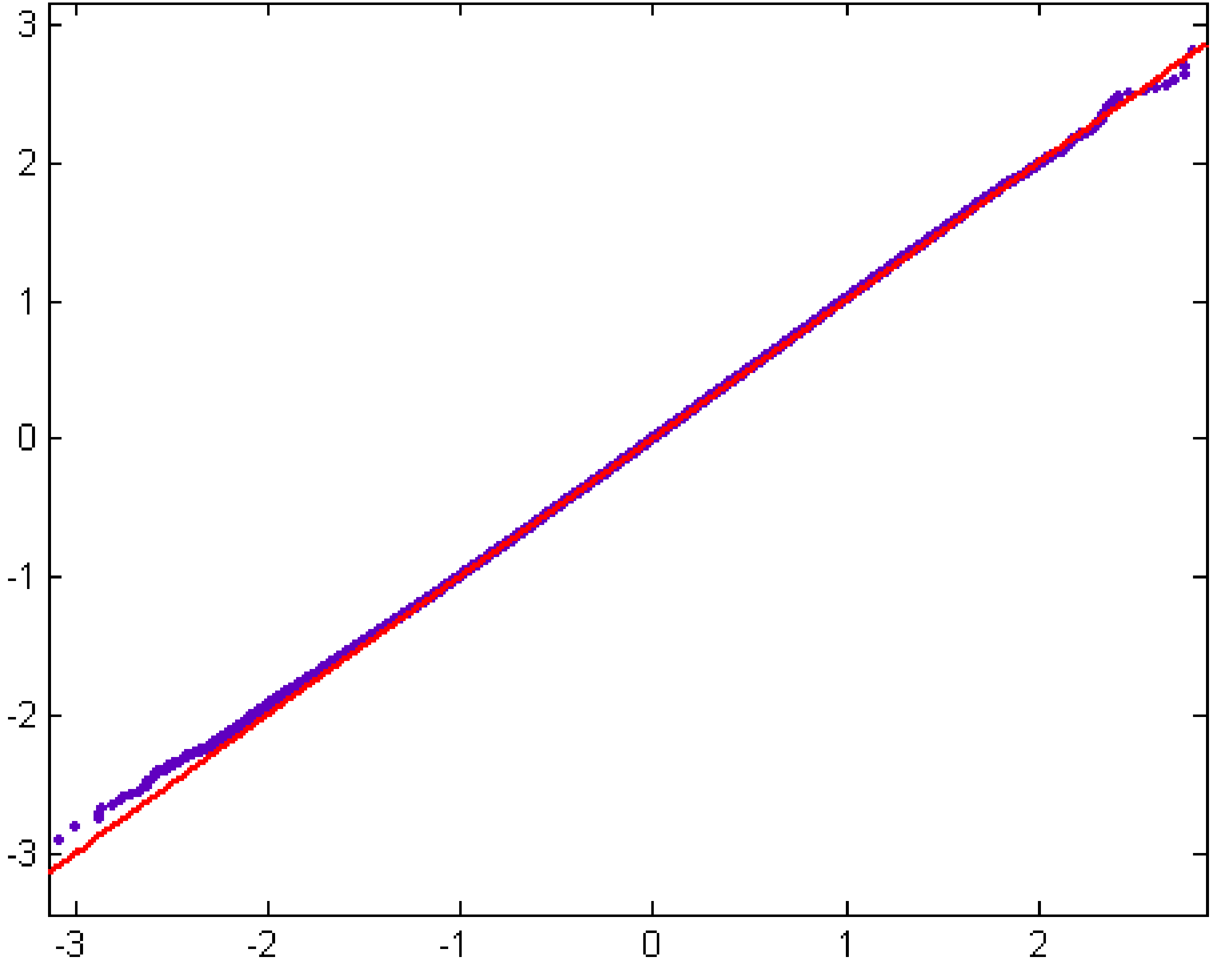


Figure 47 (n=p3q2rs, mean=0.1015, standard deviation=1.3283, size=6913)
4
2
0
-2
-4
-6
-6
-5
-4
-3
-2
-1
0
1
2
3
4

Figure 48 (n=pq, mean=-0.0562, standard deviation=0.6487 , size=188260)



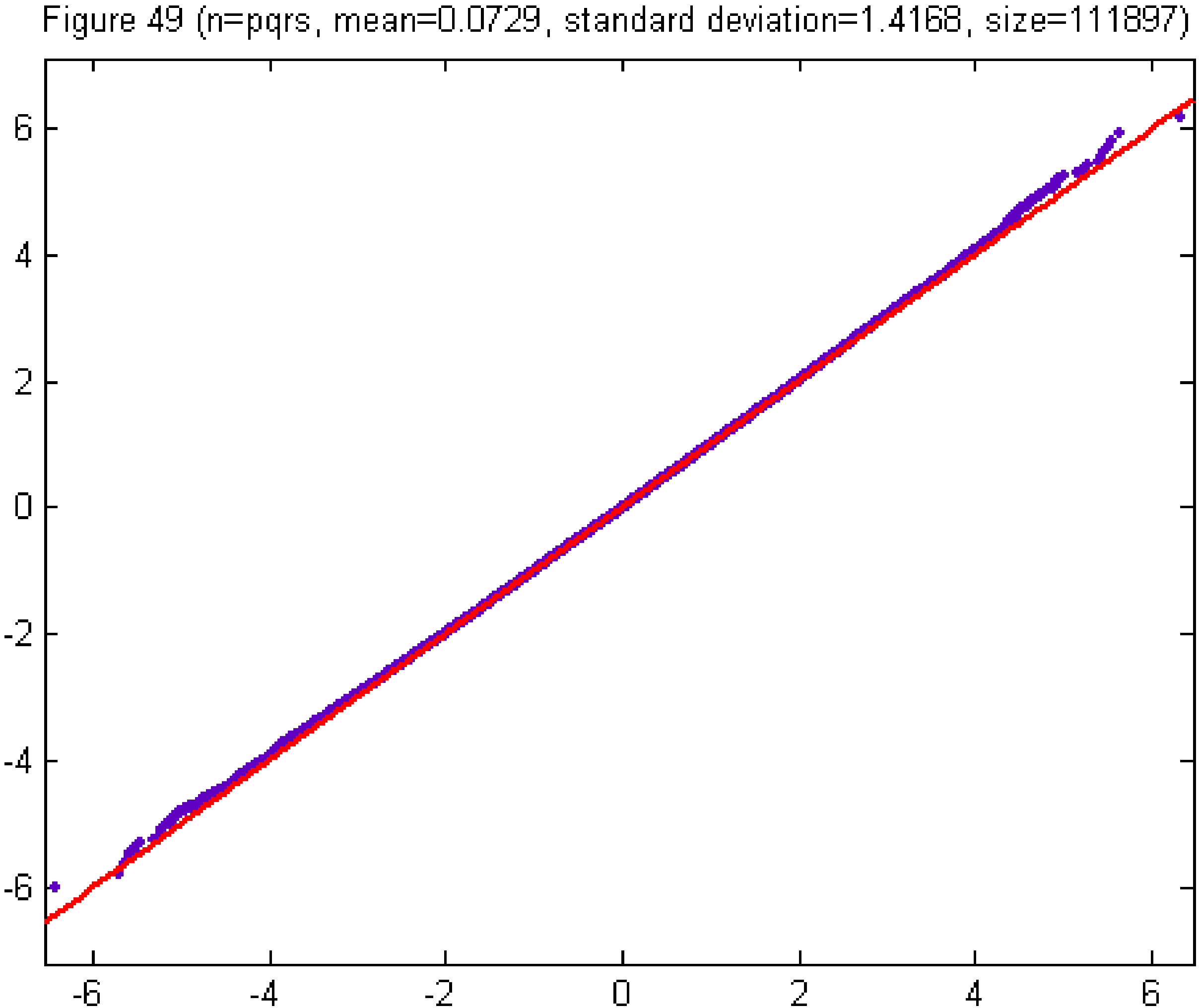


Figure 49 (n=pqrs, mean=0.0729, standard deviation=1.4168, size=111897)
6
4
2
0
-2
-4
-6
-6
-4
-2
0
2
4
6

Figure 50 ($n=p^4qr$, mean=0.1020, standard deviation=0.9257 , size=13546)

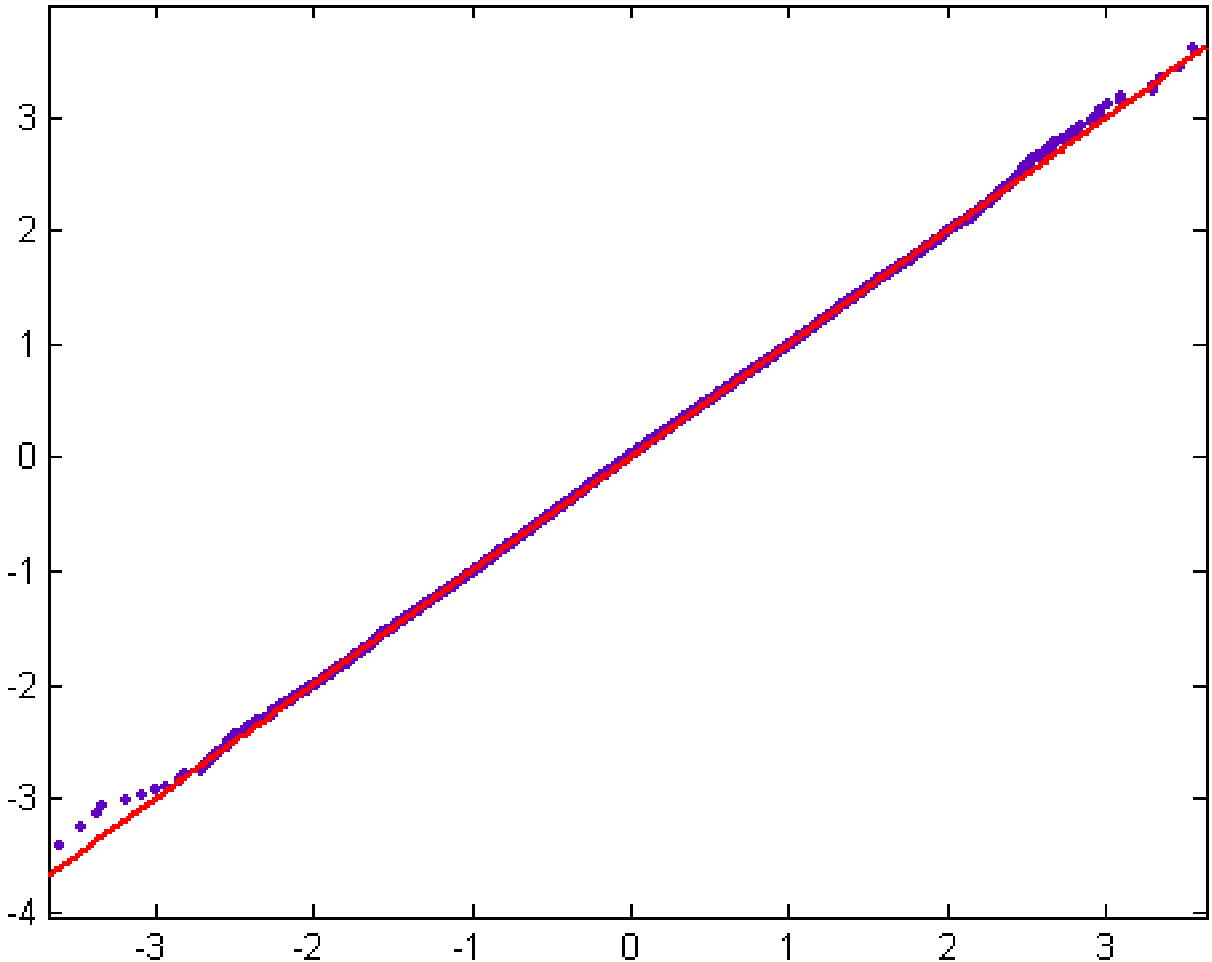
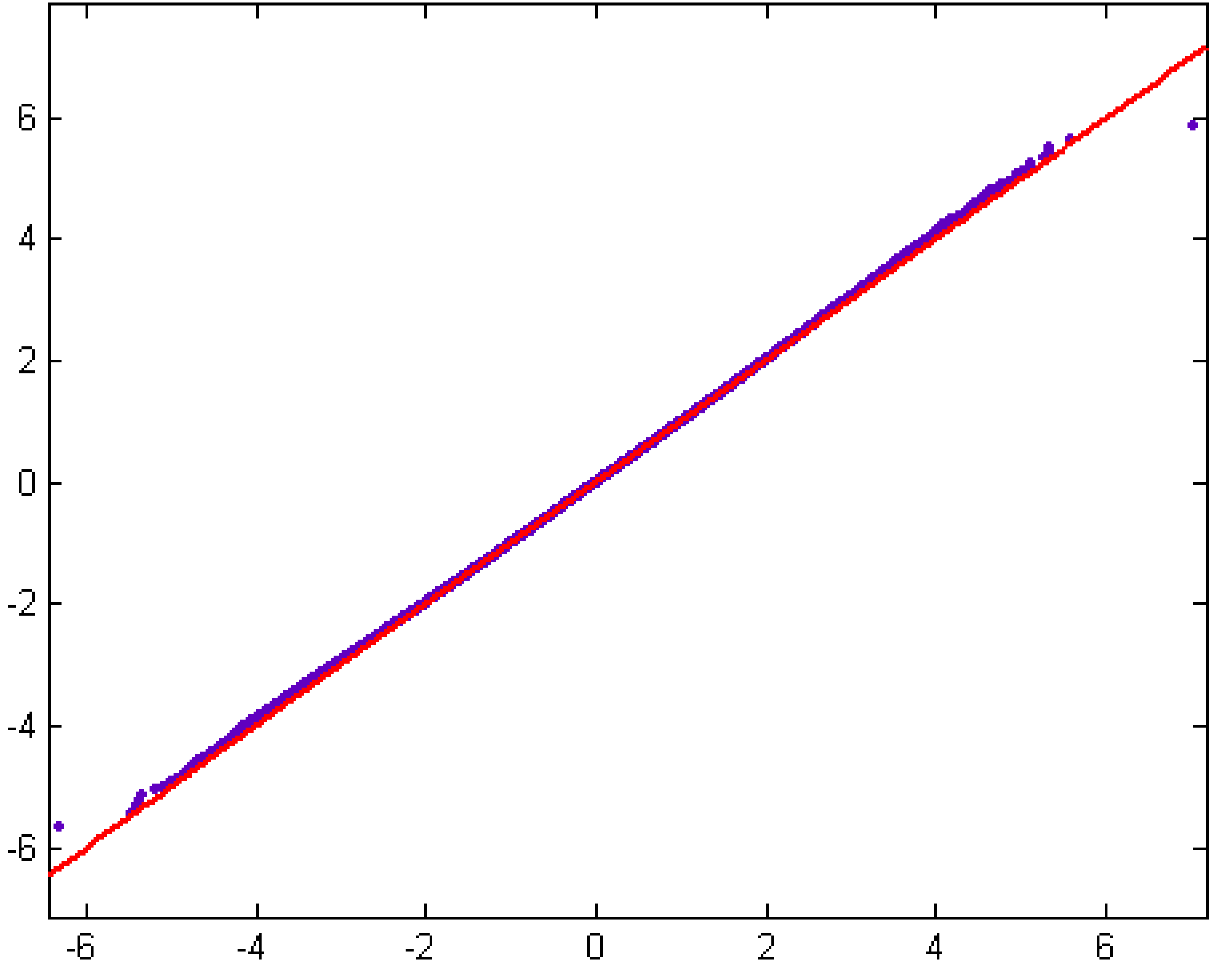


Figure 51 ($n=p^2qrs$, mean=0.0948, standard deviation=1.3852, size=63148)



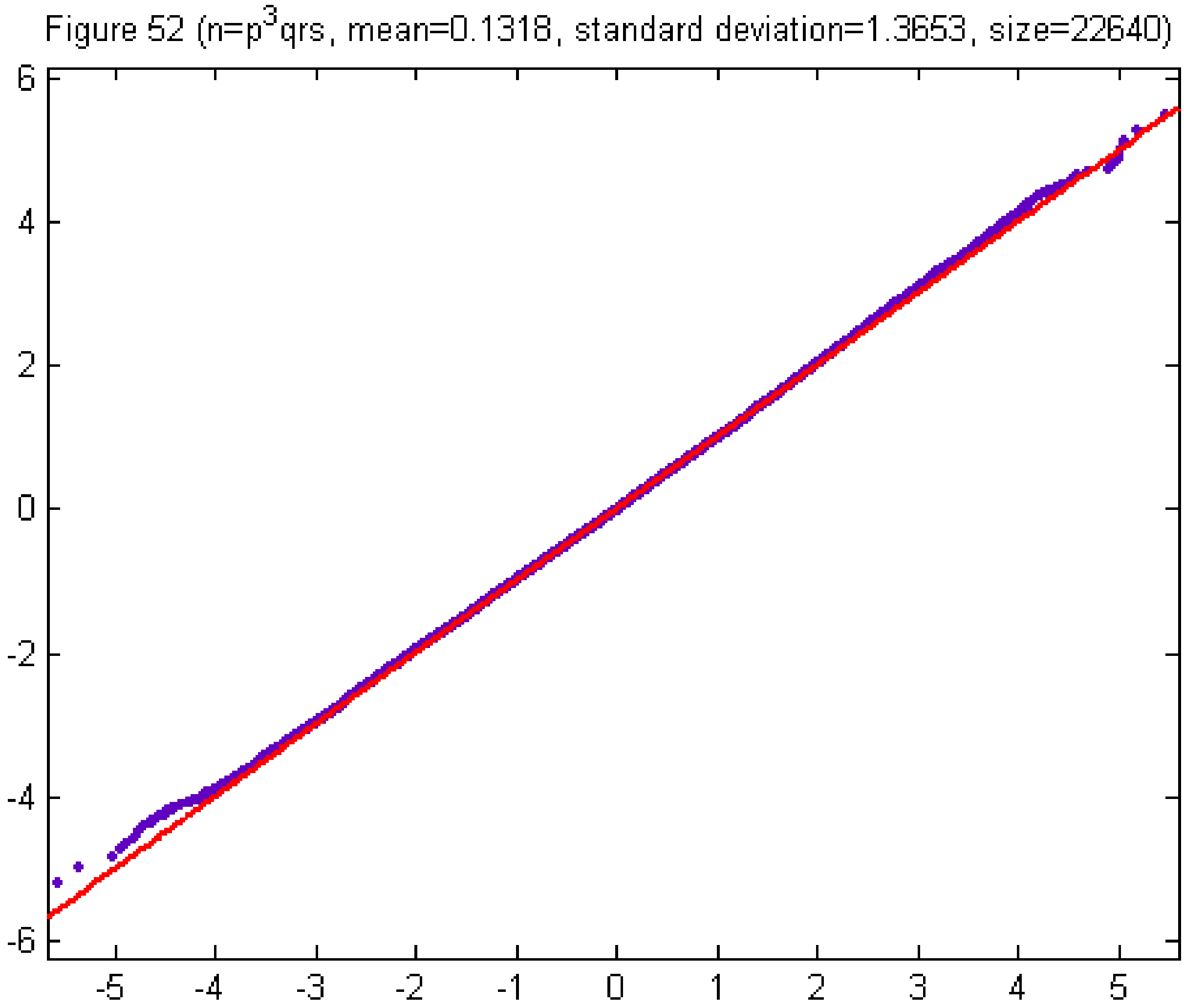
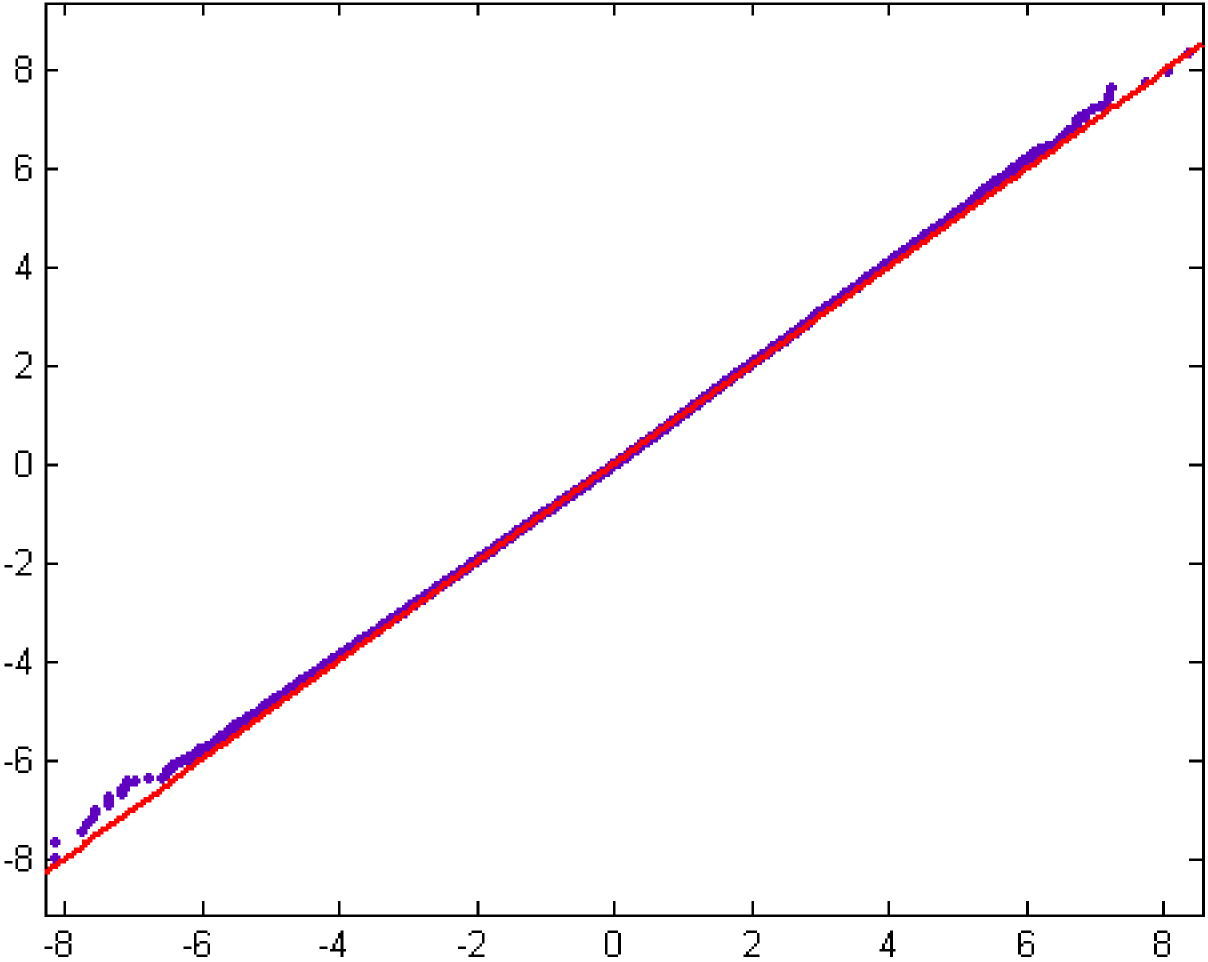
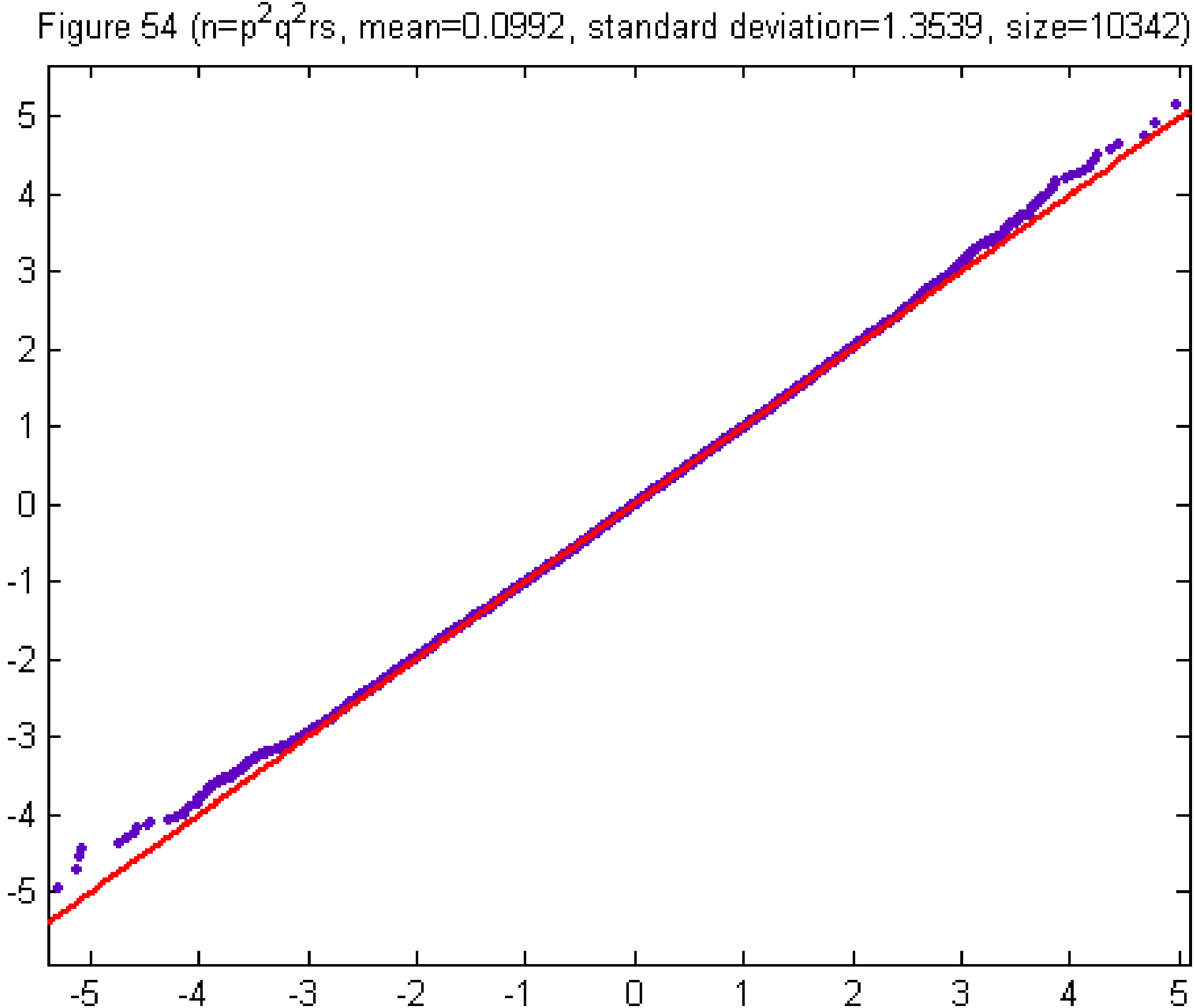


Figure 52 (n=p3qrs, mean=0.1318, standard deviation=1.3653, size=22640)
6
4
2
0
-2
-4
-6
-5
-4
-3
-2
-1
0
1
2
3
4
5

Figure 53 (n=pqrst, mean=0.1464, standard deviation=2.0514, size=29655)





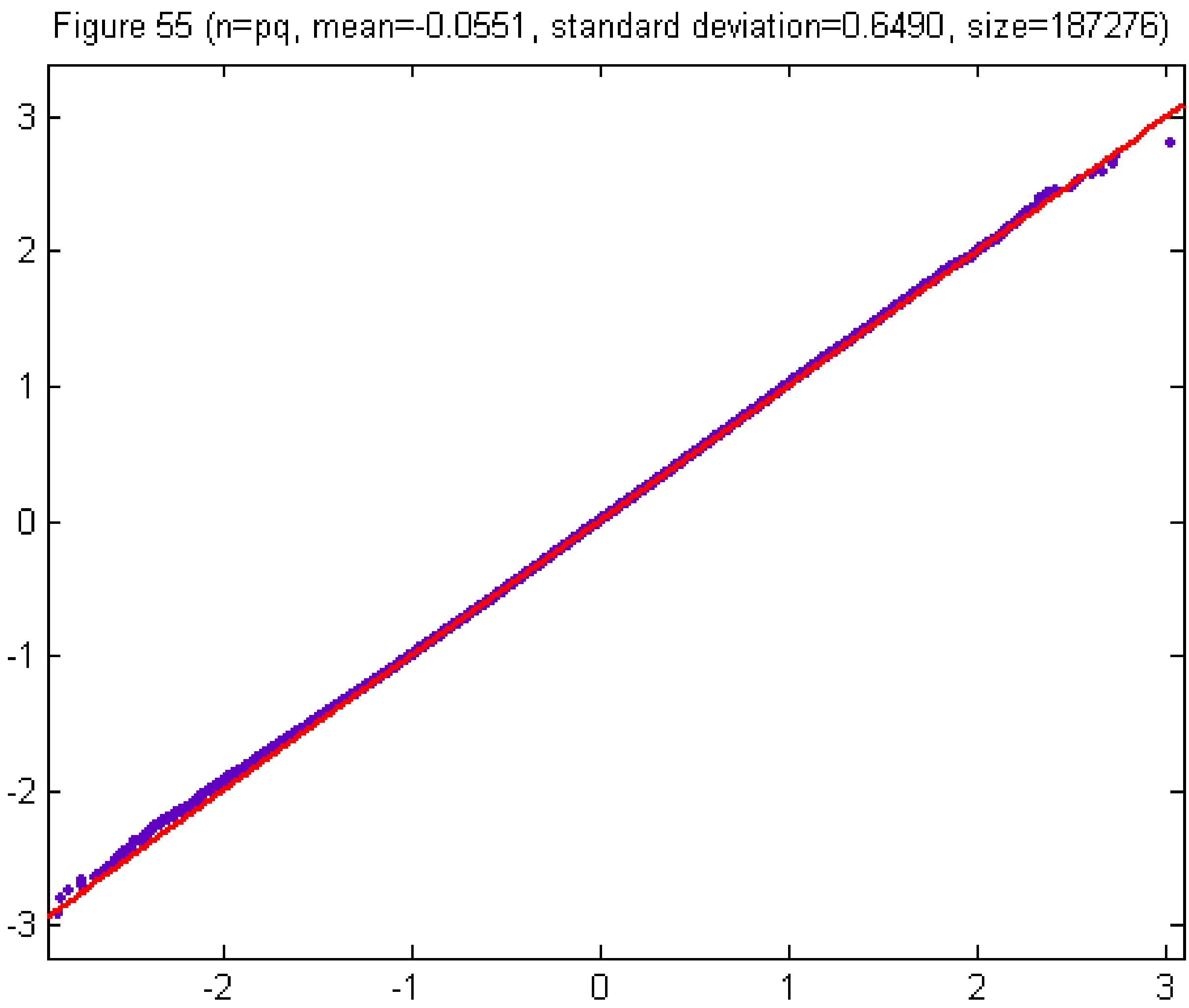


Figure 55 (n=pq, mean=-0.0551, standard deviation=0.6490, size=187276)
3
2
1
0
-1
-2
-3
-2
-1
0
1
2
3

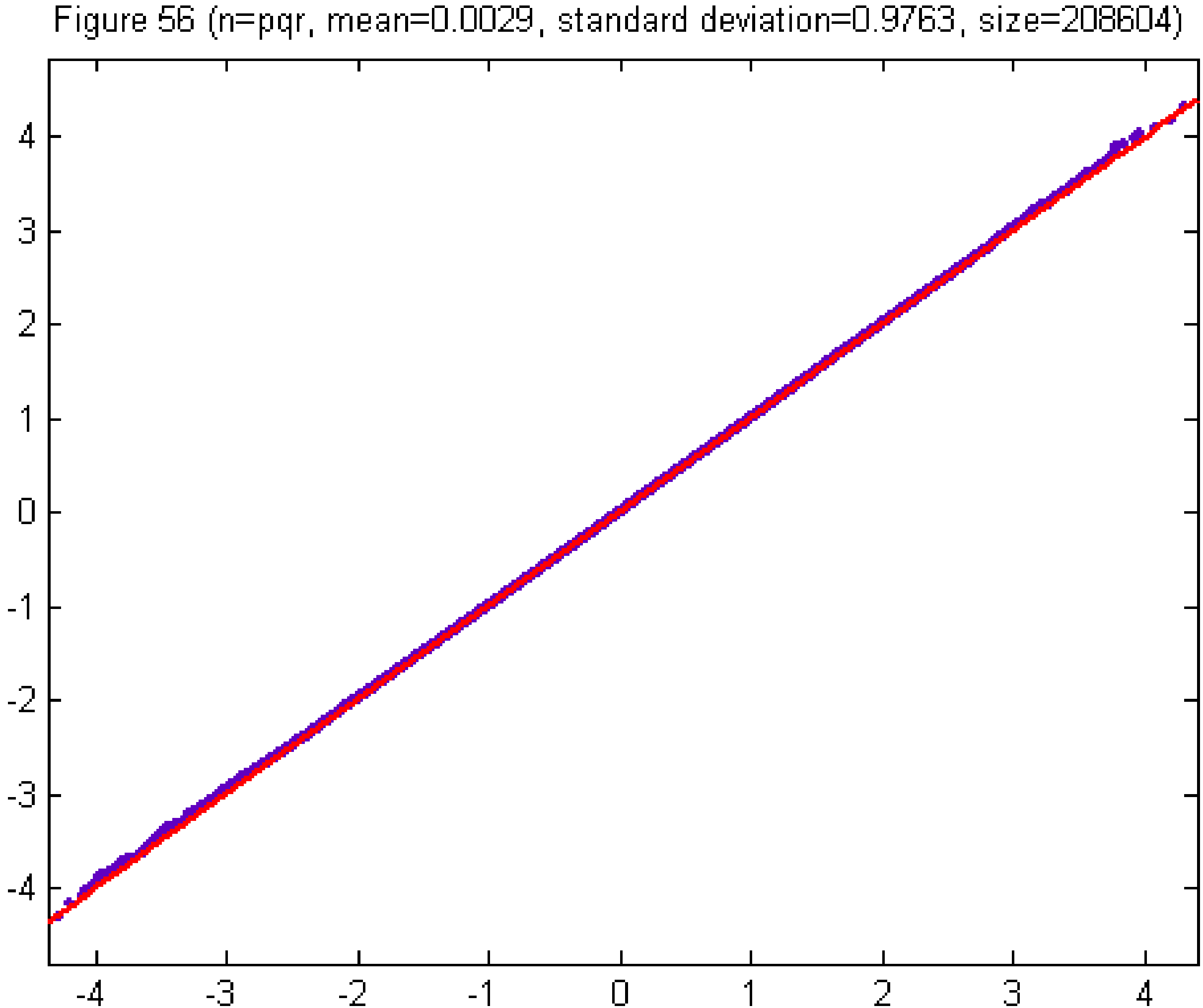
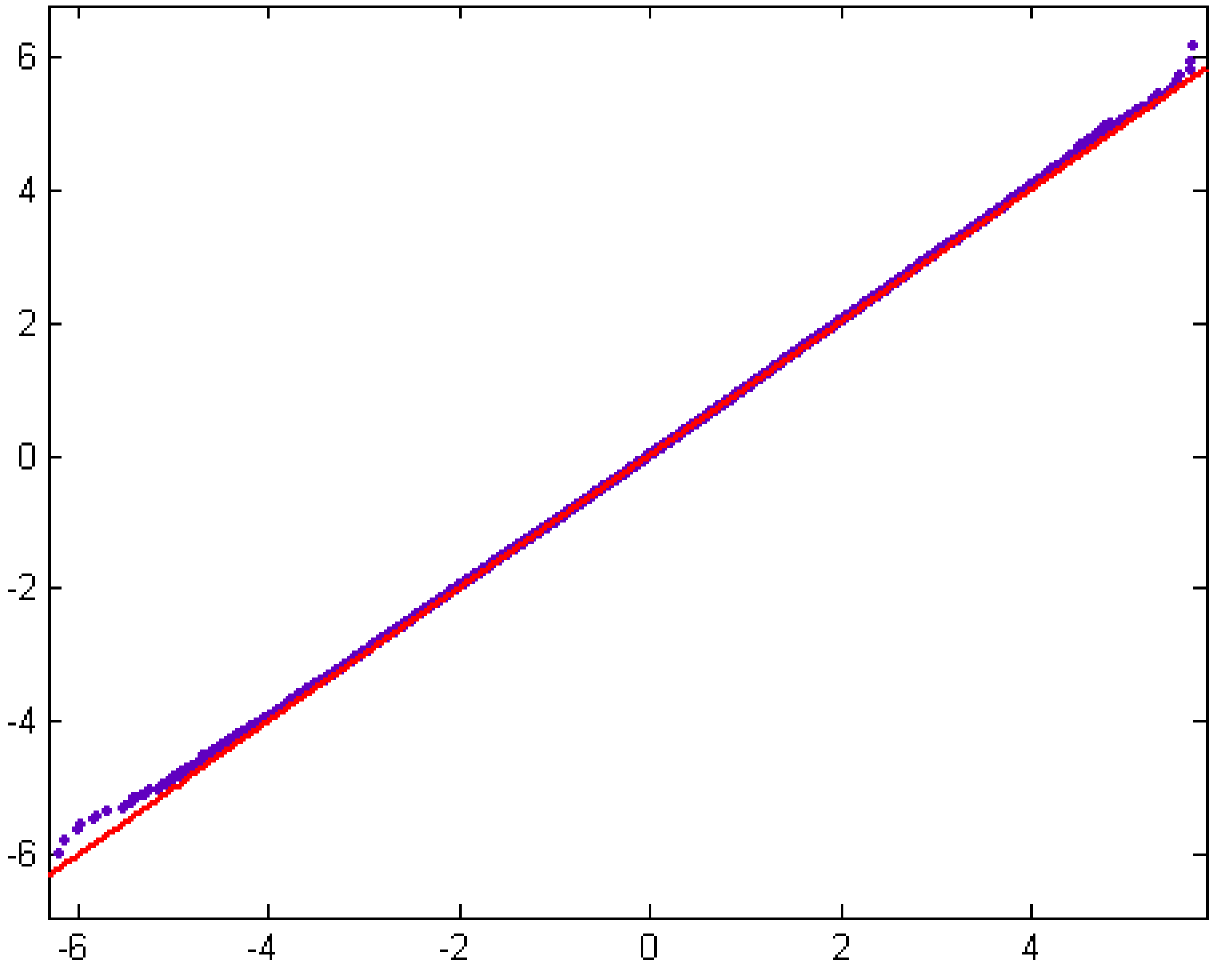


Figure 56 (n=pqr, mean=0.0029, standard deviation=0.9763, size=208604)
4
3
2
1
0
-1
-2
-3
-4
-4
-3
-2
-1
0
1
2
3
4

Figure 57 (n=pqrs, mean=0.0730, standard deviation=1.4160, size=113555)



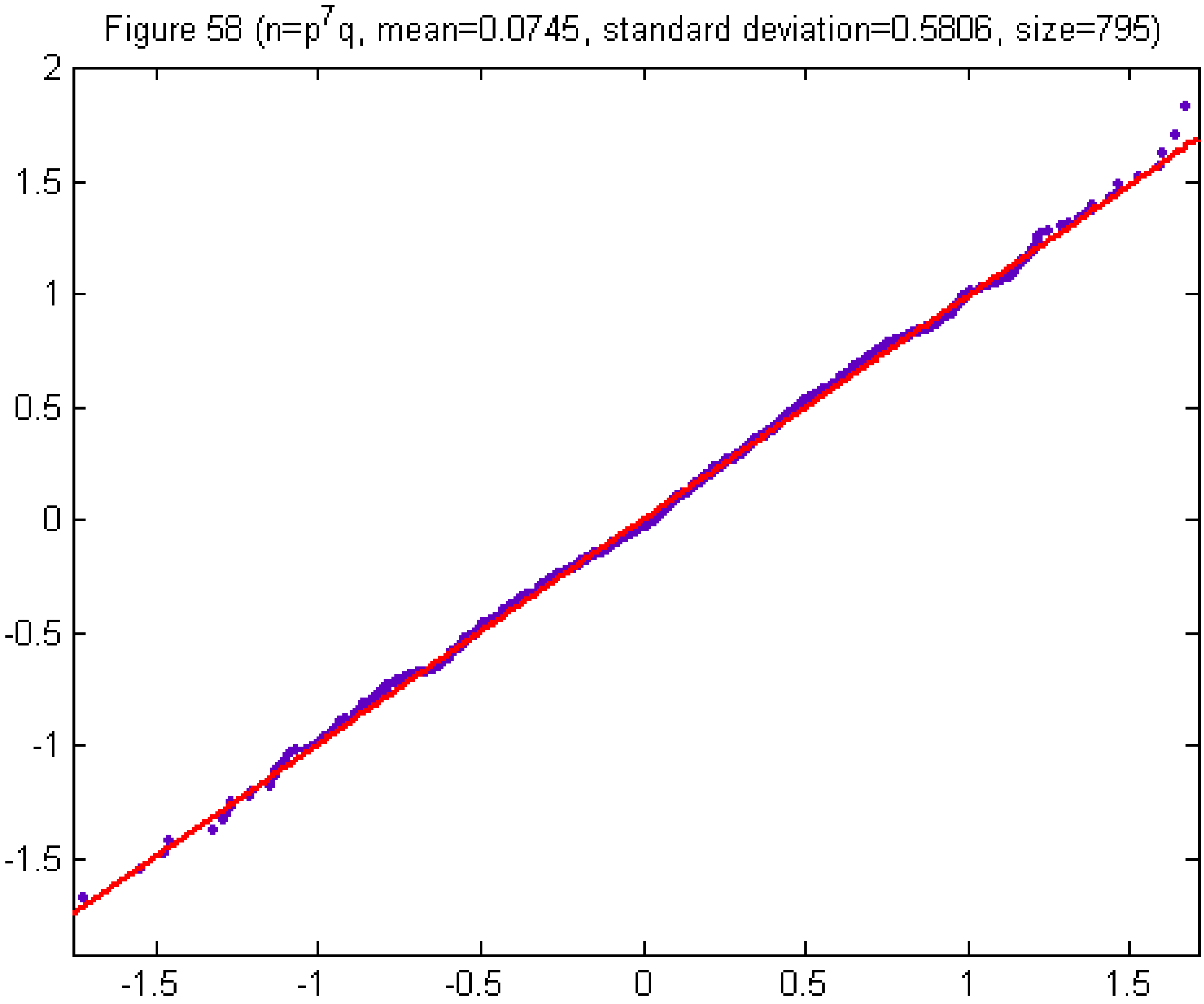
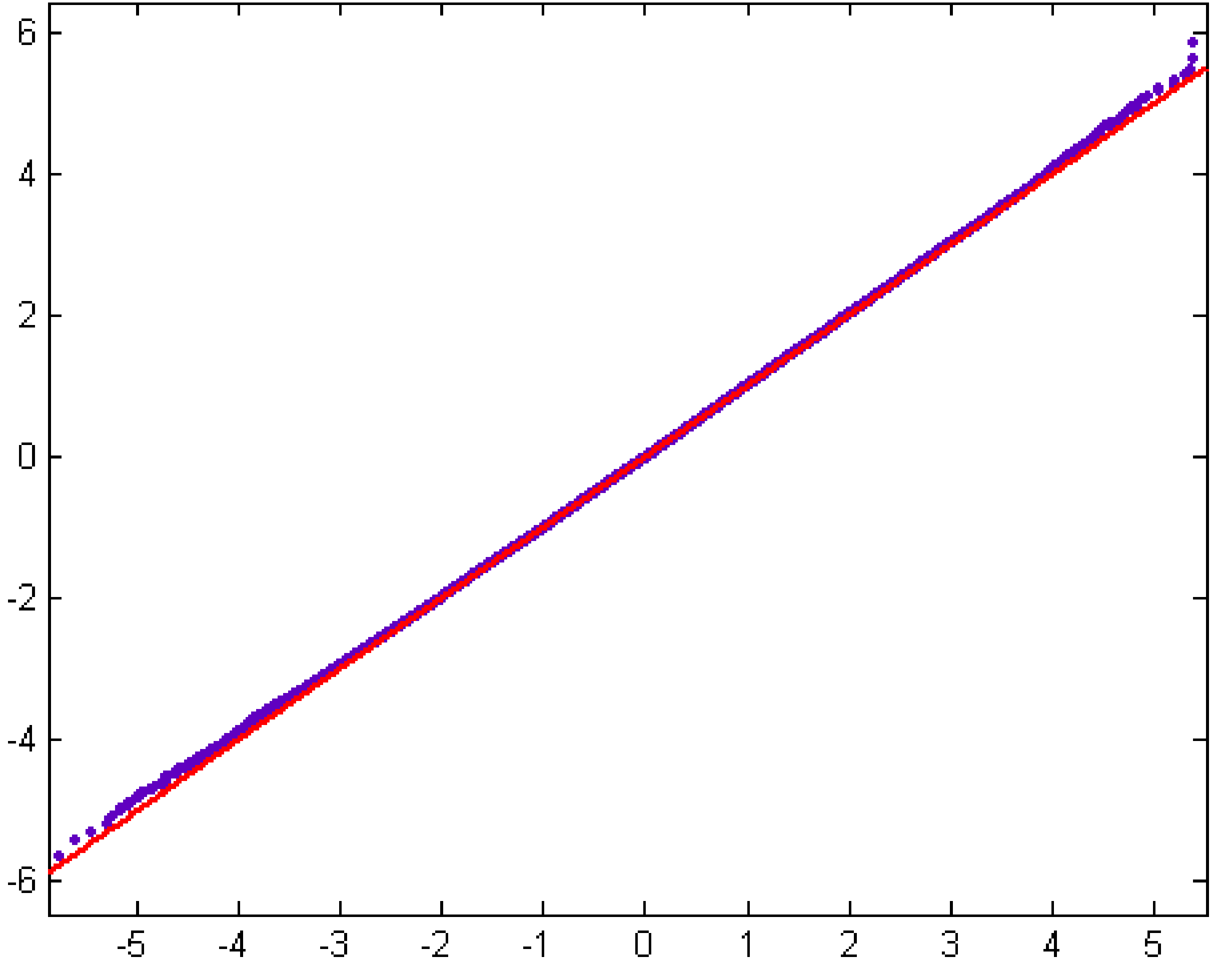


Figure 58 (n=p^7q, mean=0.0745, standard deviation=0.5806, size=795)
2
1.5
1
0.5
0
-0.5
-1
-1.5
-1.5
-1
-0.5
0
0.5
1
1.5

Figure 59 ($n=p^2qrs$, mean=0.0971, standard deviation=1.3845, size=63377)



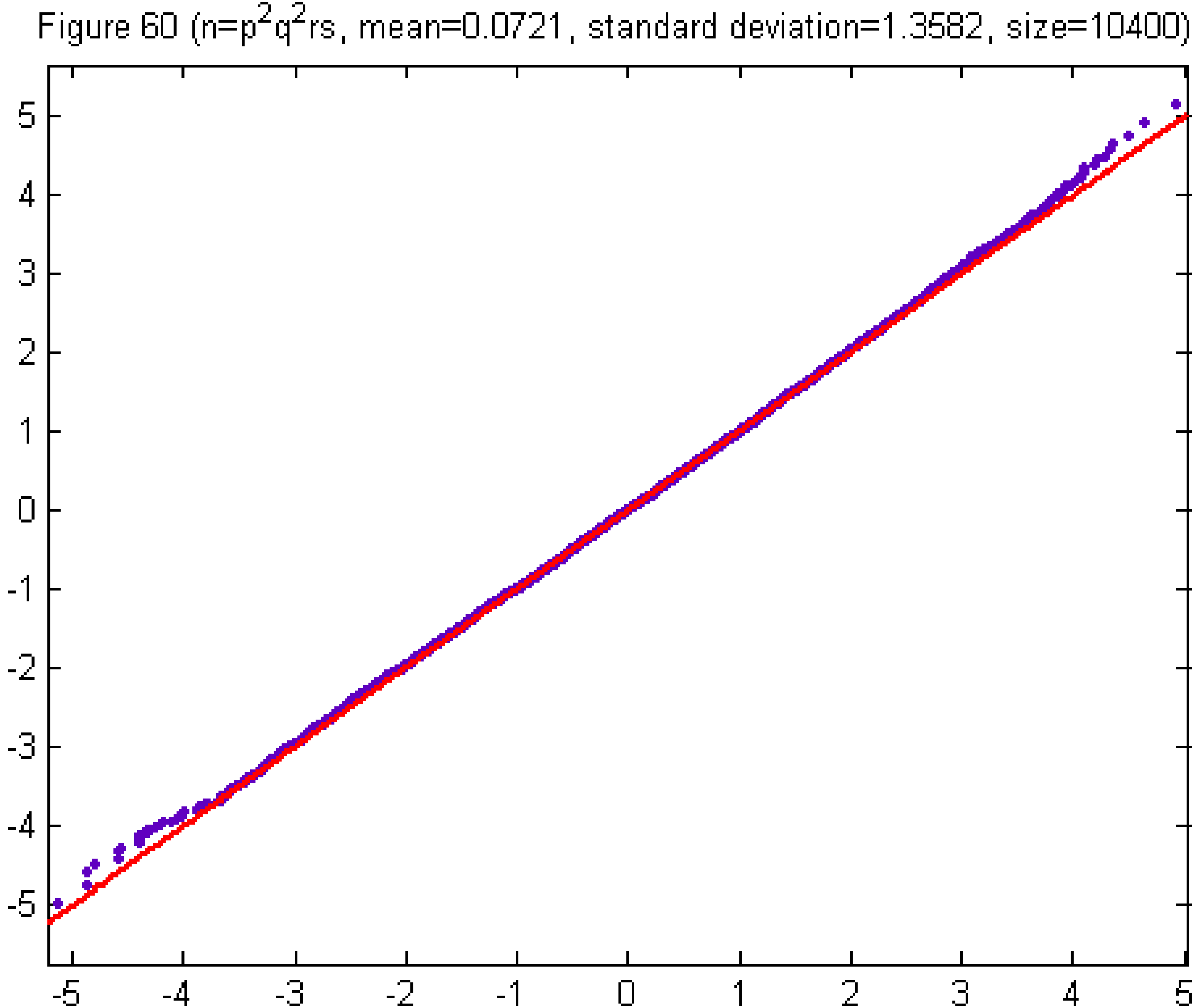
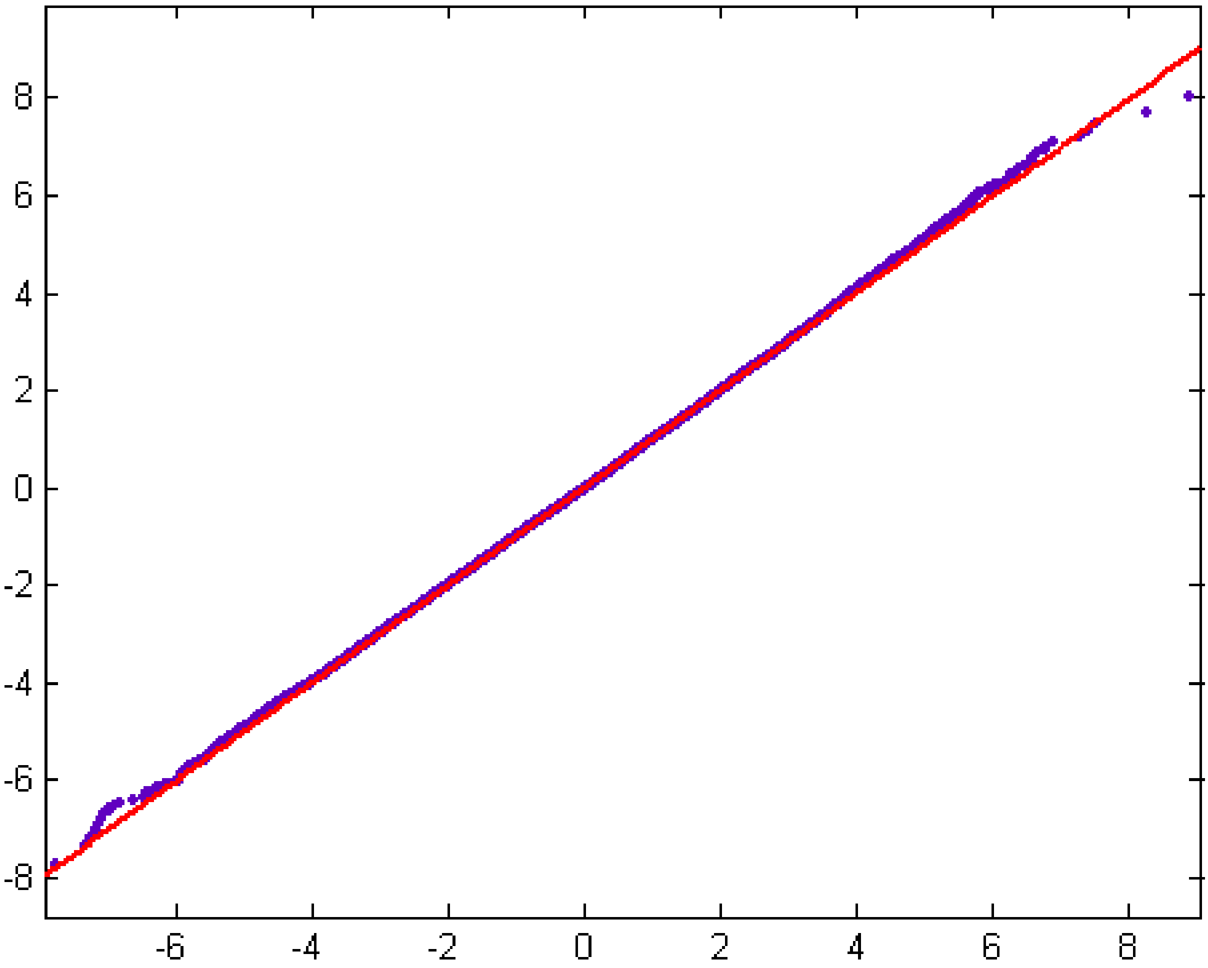


Figure 60 (n=p2q2rs, mean=0.0721, standard deviation=1.3582, size=10400)
5
4
3
2
1
0
-1
-2
-3
-4
-5
-5
-4
-3
-2
-1
0
1
2
3
4
5

Figure 61 (n=p^2qrst, mean=0.1505, standard deviation=2.0132, size=21108)



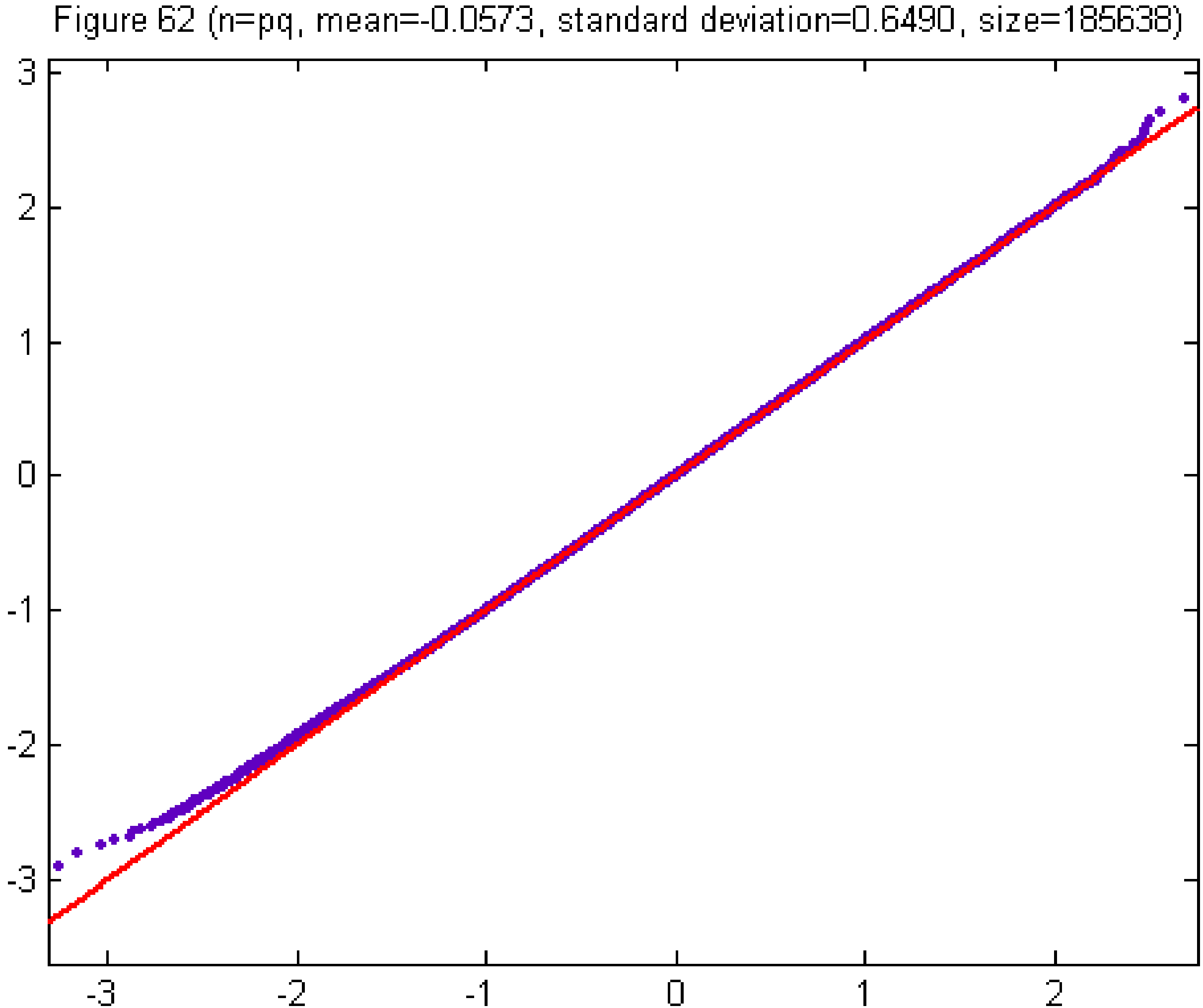


Figure 62 (n=pq, mean=-0.0573, standard deviation=0.6490, size=185638)
3
2
1
0
-1
-2
-3
-3
-2
-1
0
1
2

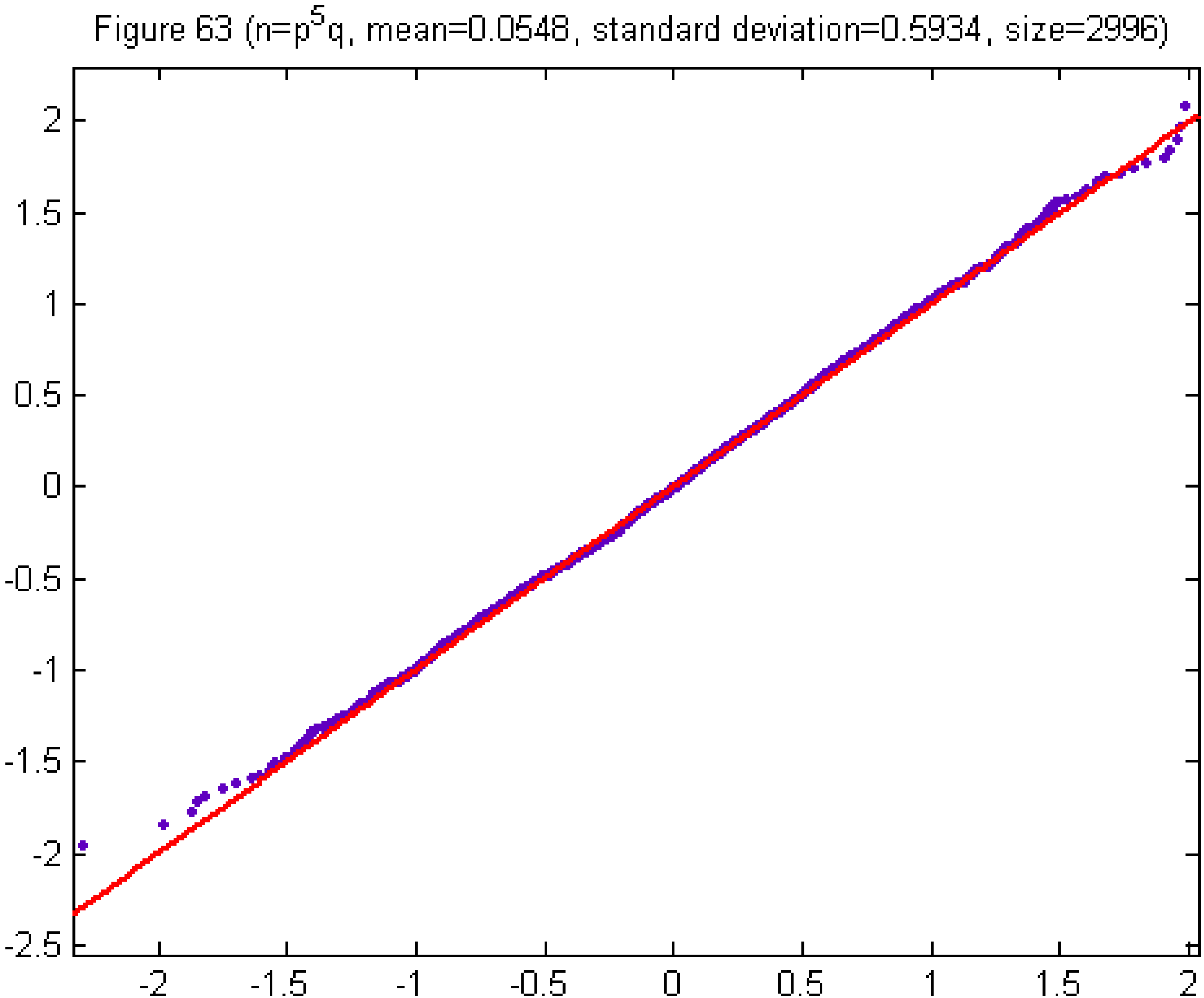


Figure 63 (n=p^5q, mean=0.0548, standard deviation=0.5934, size=2996)
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
-2.5
-2
-1.5
-1
-0.5
0
0.5
1
1.5
2

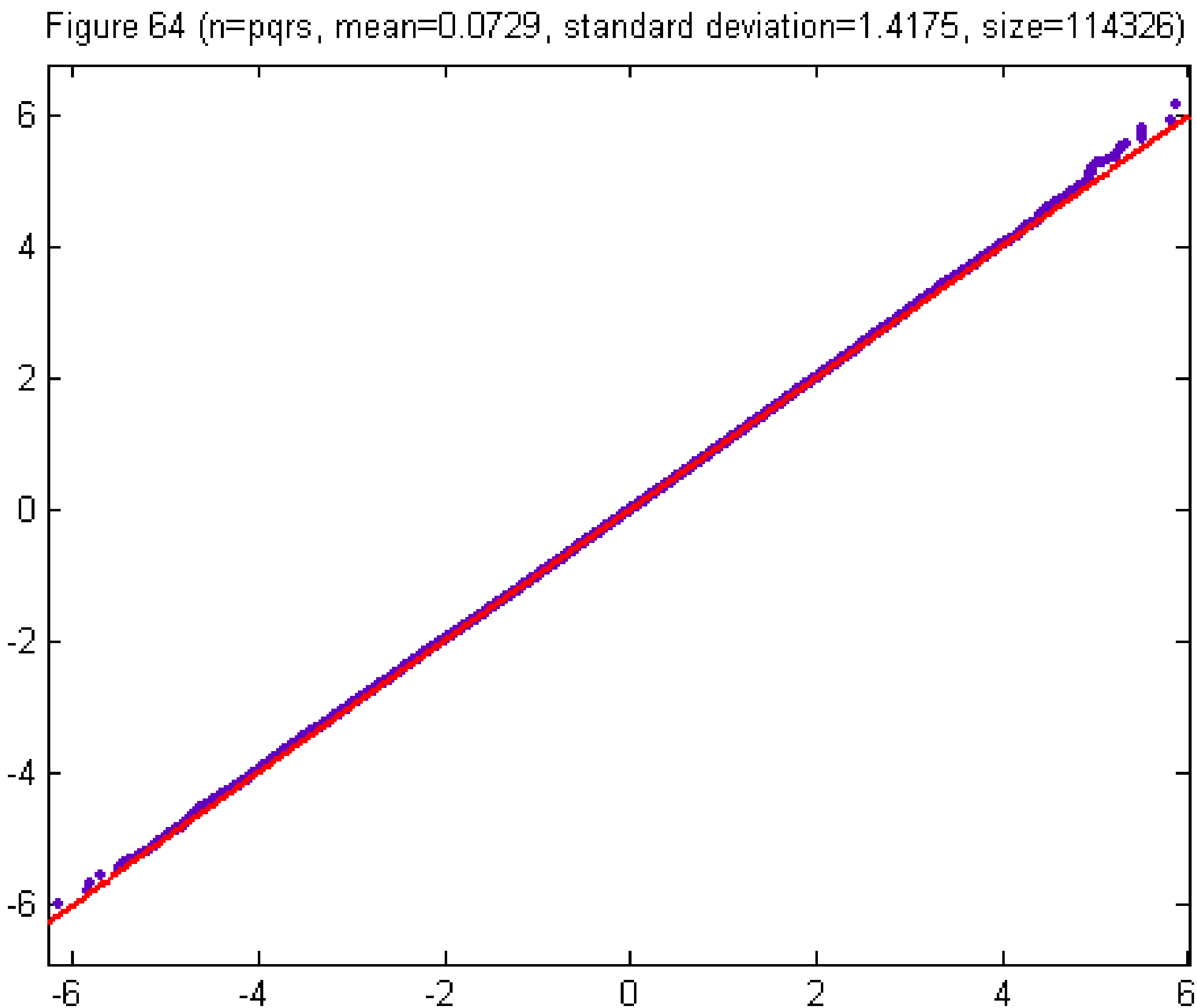


Figure 64 (n=pqrs, mean=0.0729, standard deviation=1.4175, size=114326)
6
4
2
0
-2
-4
-6
-6
-4
-2
0
2
4
6

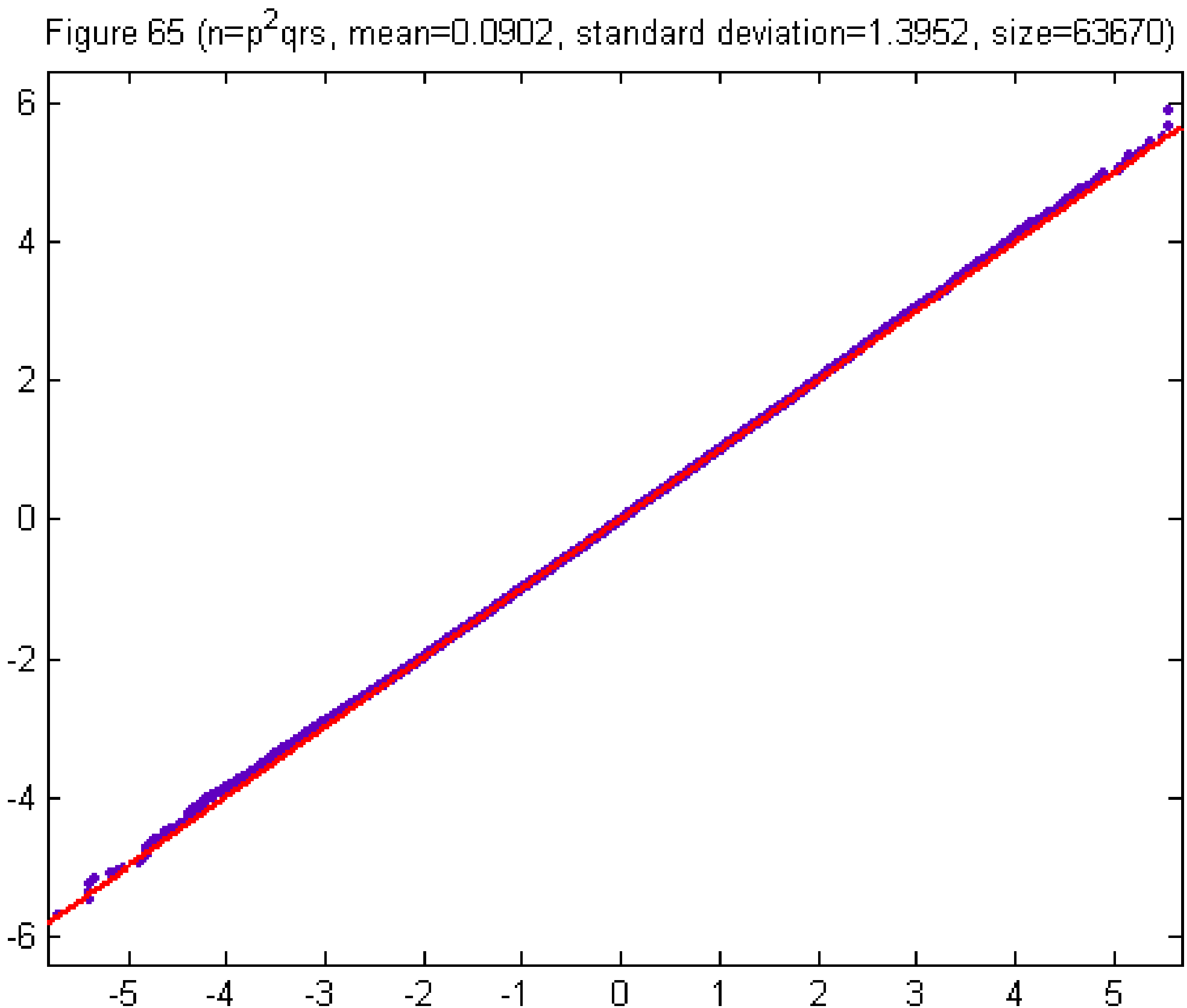
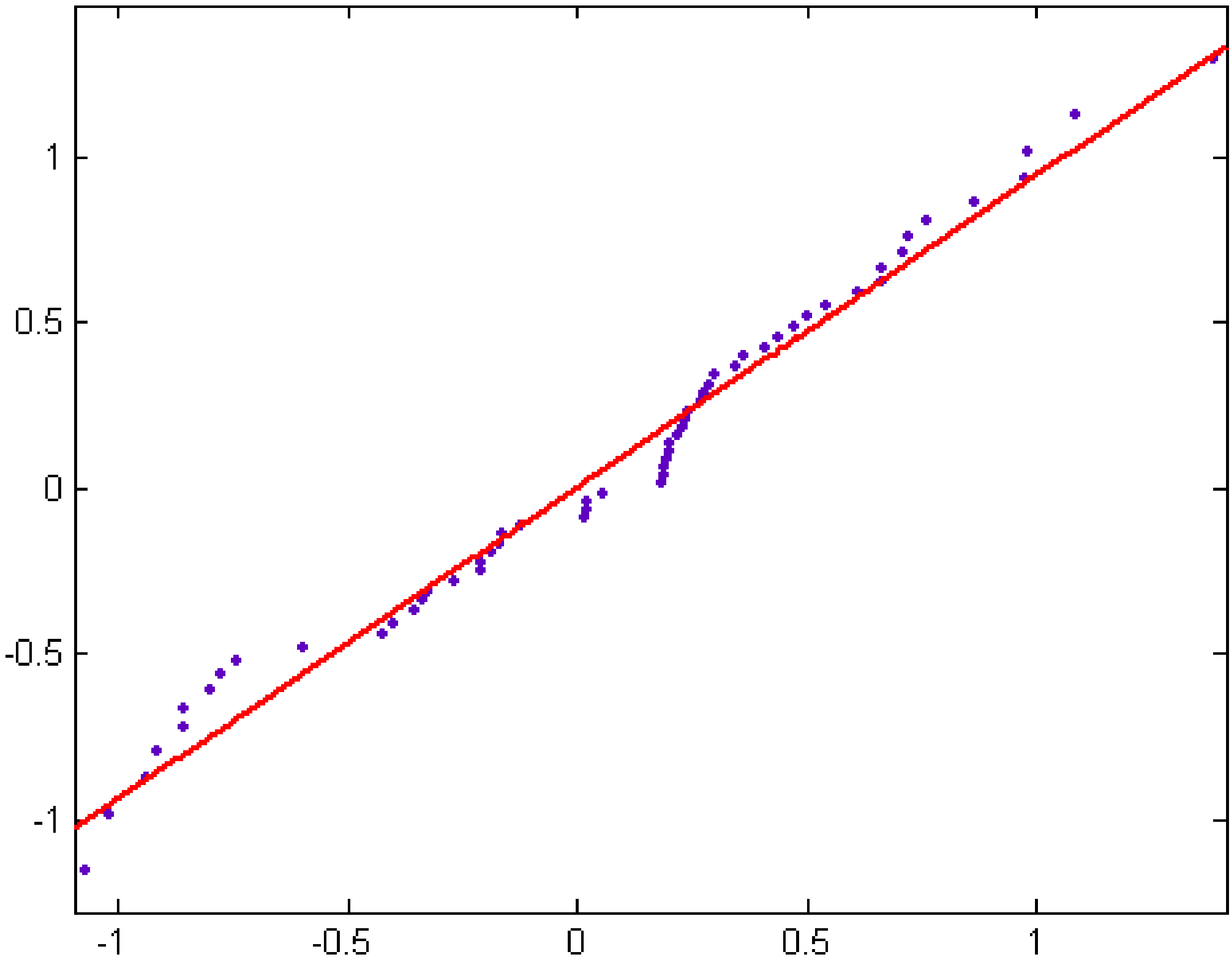


Figure 65 (n=p²qrs, mean=0.0902, standard deviation=1.3952, size=63670)
6
4
2
0
-2
-4
-6
-5
-4
-3
-2
-1
0
1
2
3
4
5

Figure 66 ($n=p^{11}q$, mean=0.0708, standard deviation=0.5771, size=58)



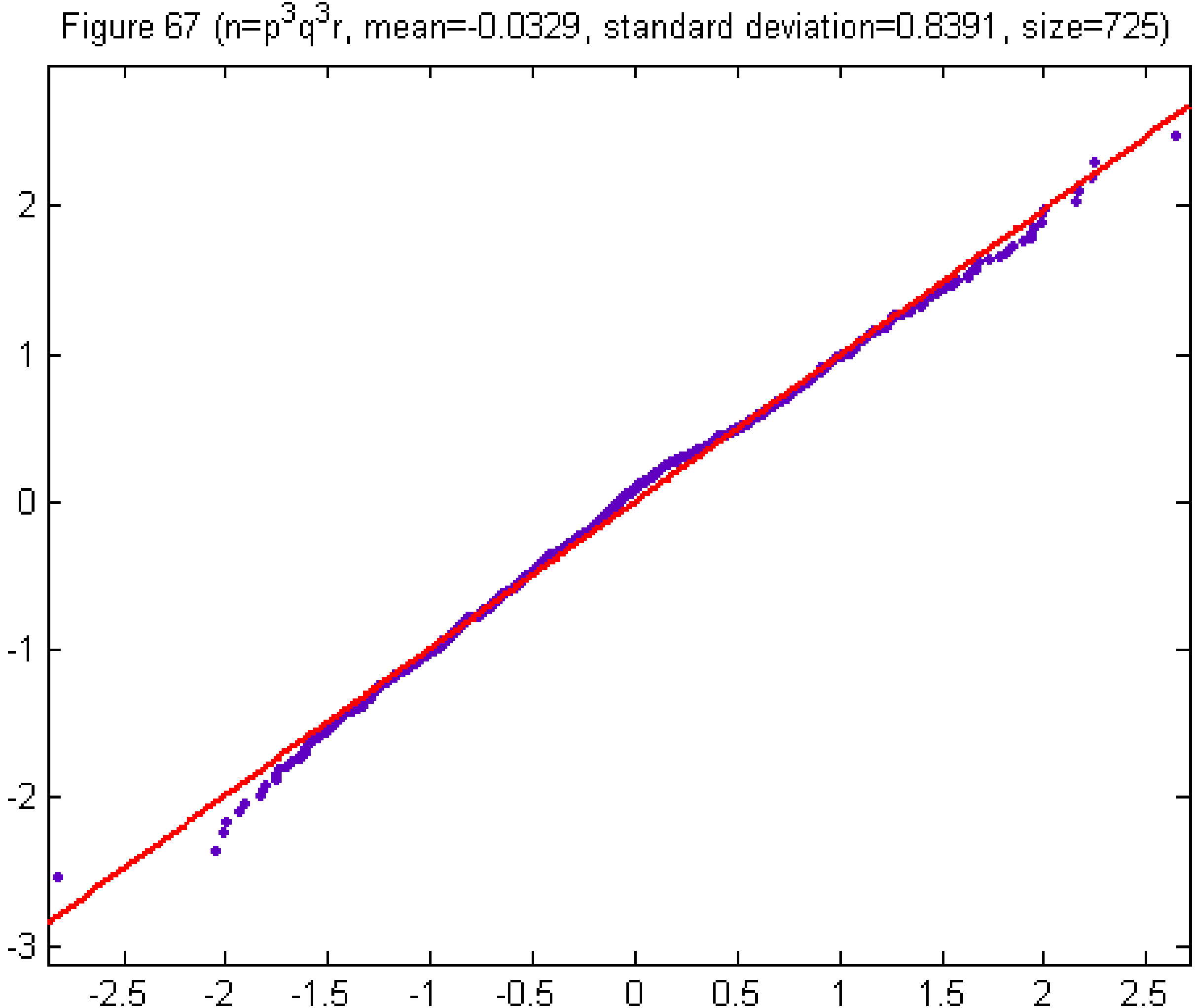


Figure 67 (n=p³q³r, mean=-0.0329, standard deviation=0.8391, size=725)
2
1
0
-1
-2
-3
-2.5
-2
-1.5
-1
-0.5
0
0.5
1
1.5
2
2.5

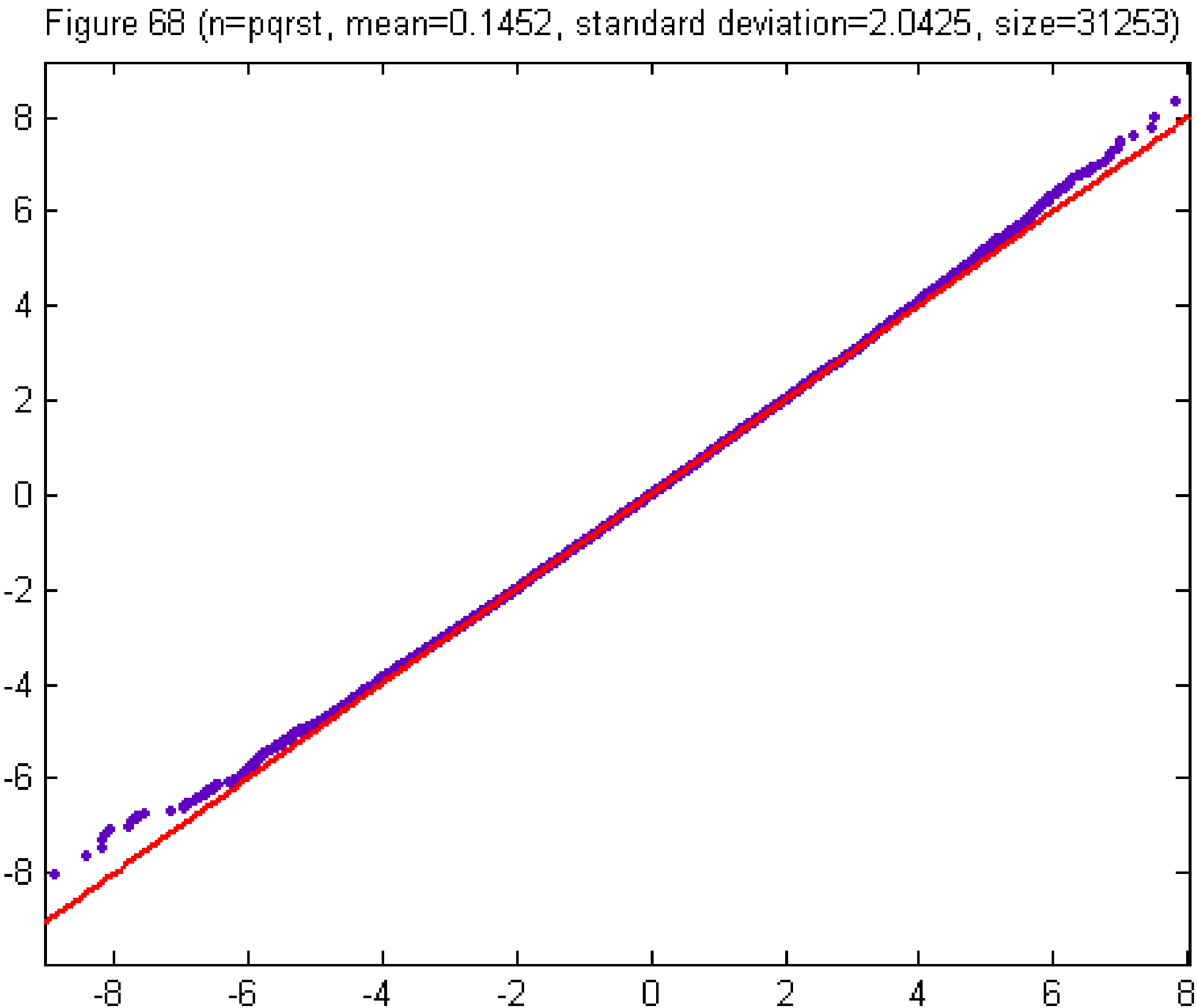


Figure 68 (n=pqrst, mean=0.1452, standard deviation=2.0425, size=31253)
8
6
4
2
0
-2
-4
-6
-8
-8
-6
-4
-2
0
2
4
6
8

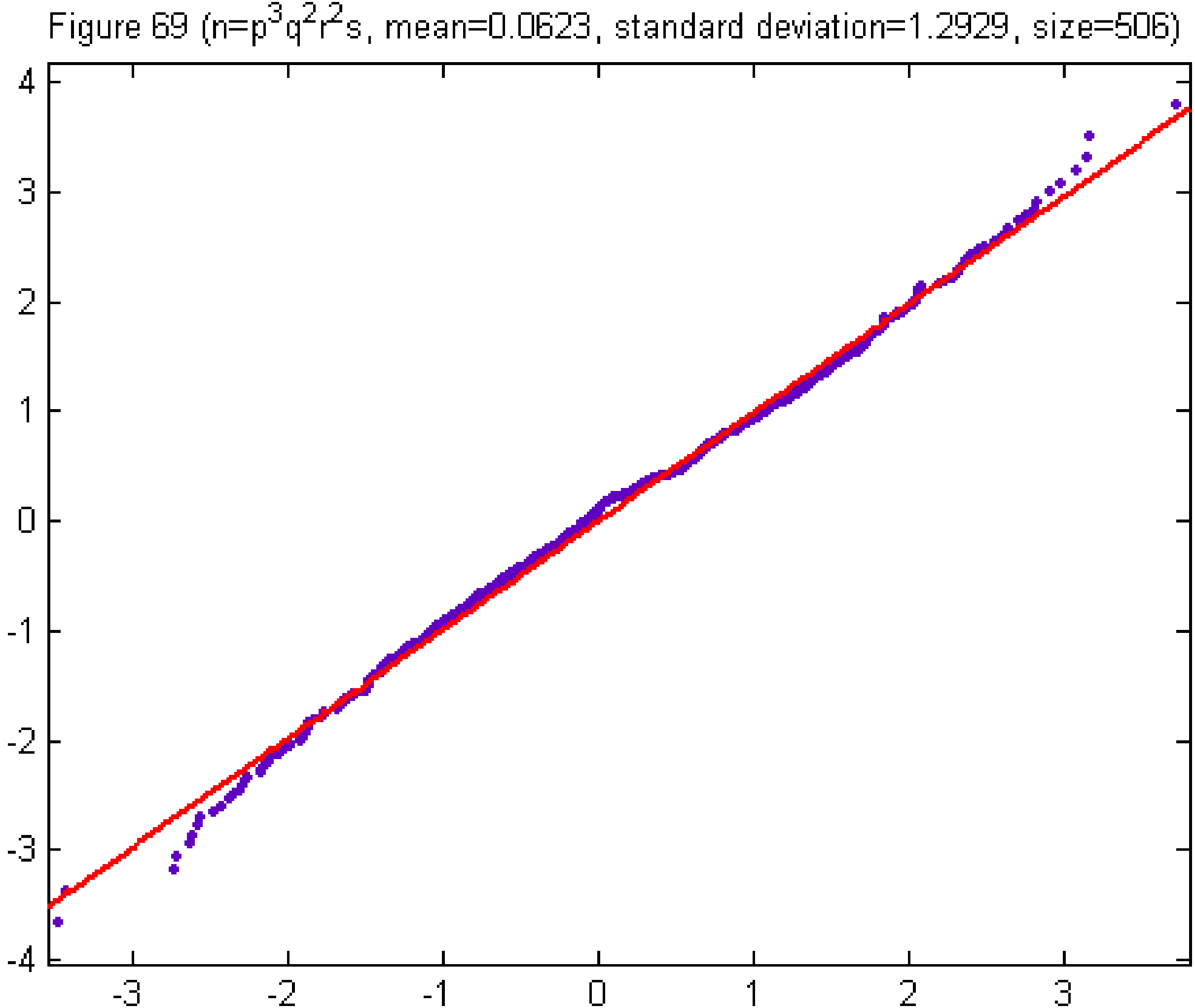


Figure 69 (n=p^3q^2r^2s, mean=0.0623, standard deviation=1.2929, size=506)
4
3
2
1
0
-1
-2
-3
-4
-3
-2
-1
0
1
2
3

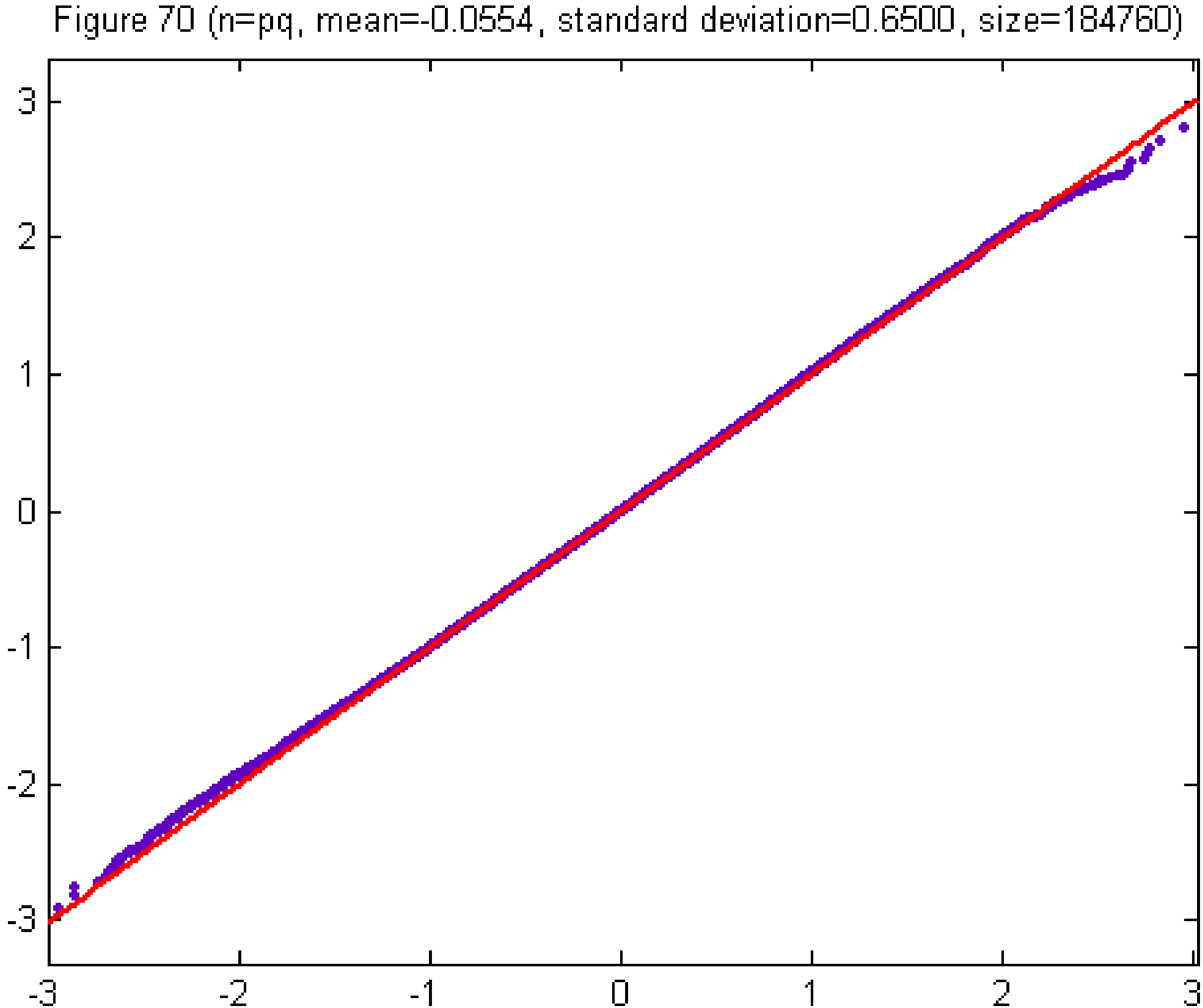


Figure 70 (n=pq, mean=-0.0554, standard deviation=0.6500, size=184760)
3
2
1
0
-1
-2
-3
-3
-2
-1
0
1
2
3

Figure 71 (n=pqr, mean=0.0031, standard deviation=0.9780, size=208784)

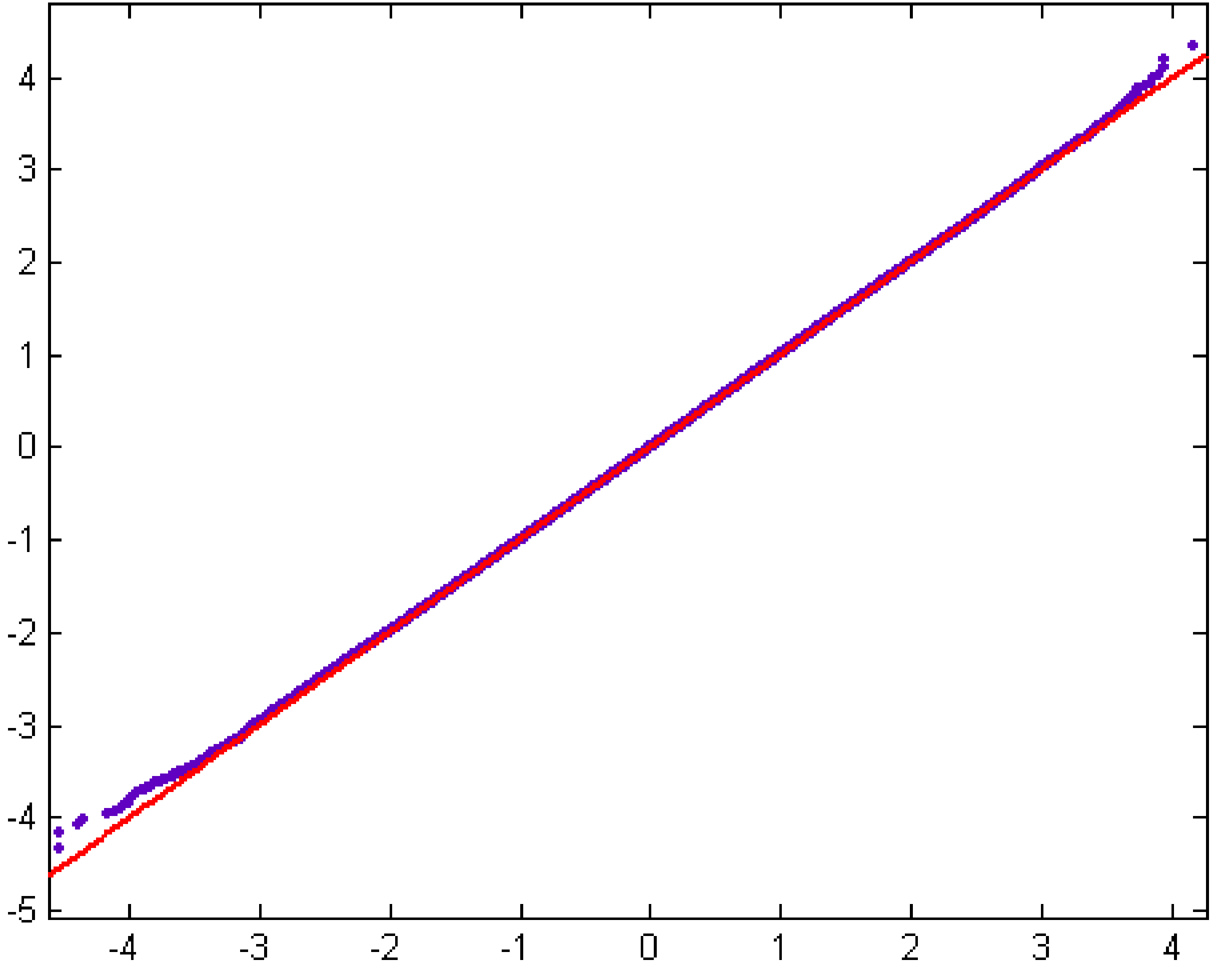


Figure 72 (n=pqrs, mean=0.0723, standard deviation=1.4202, size=115239)

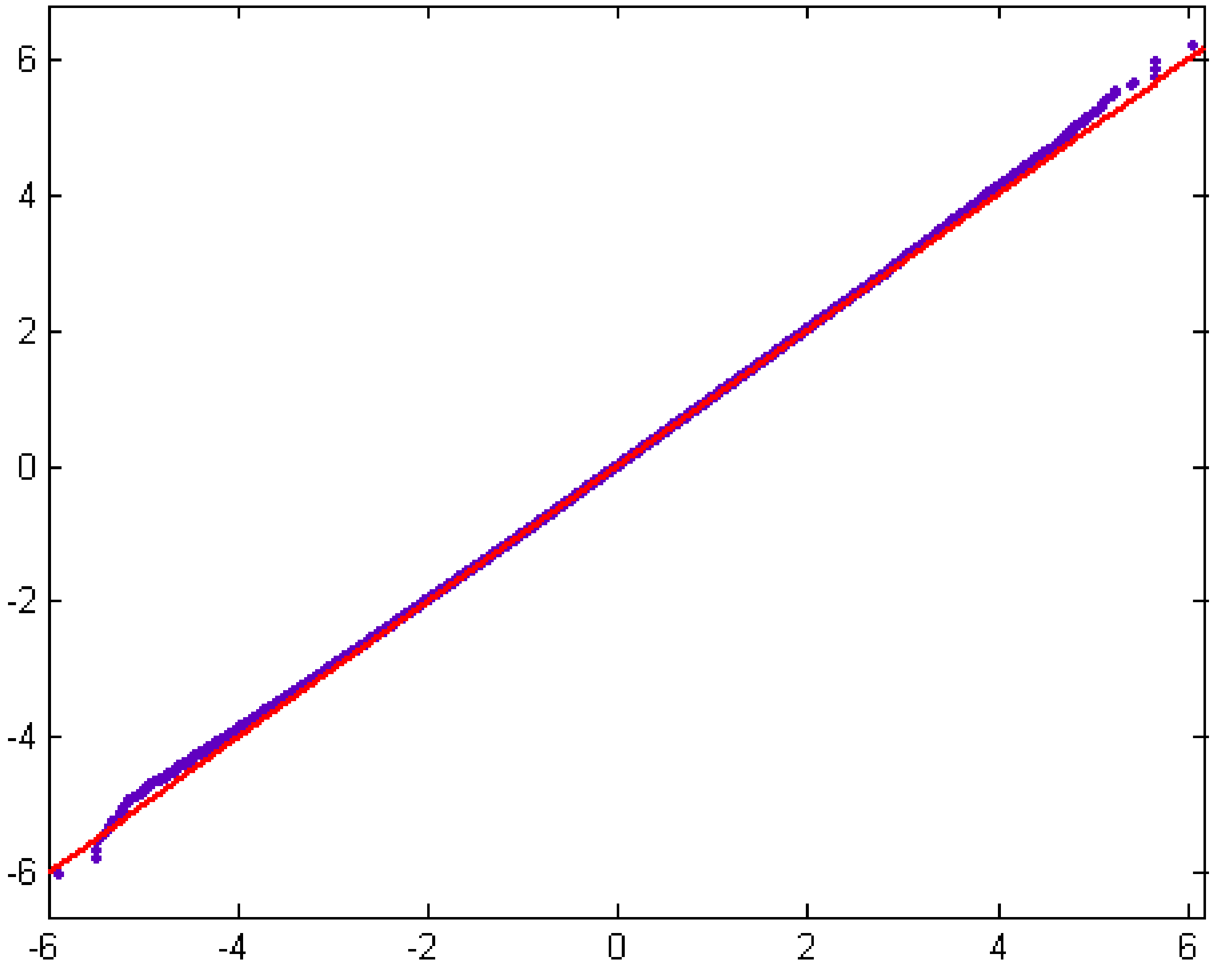
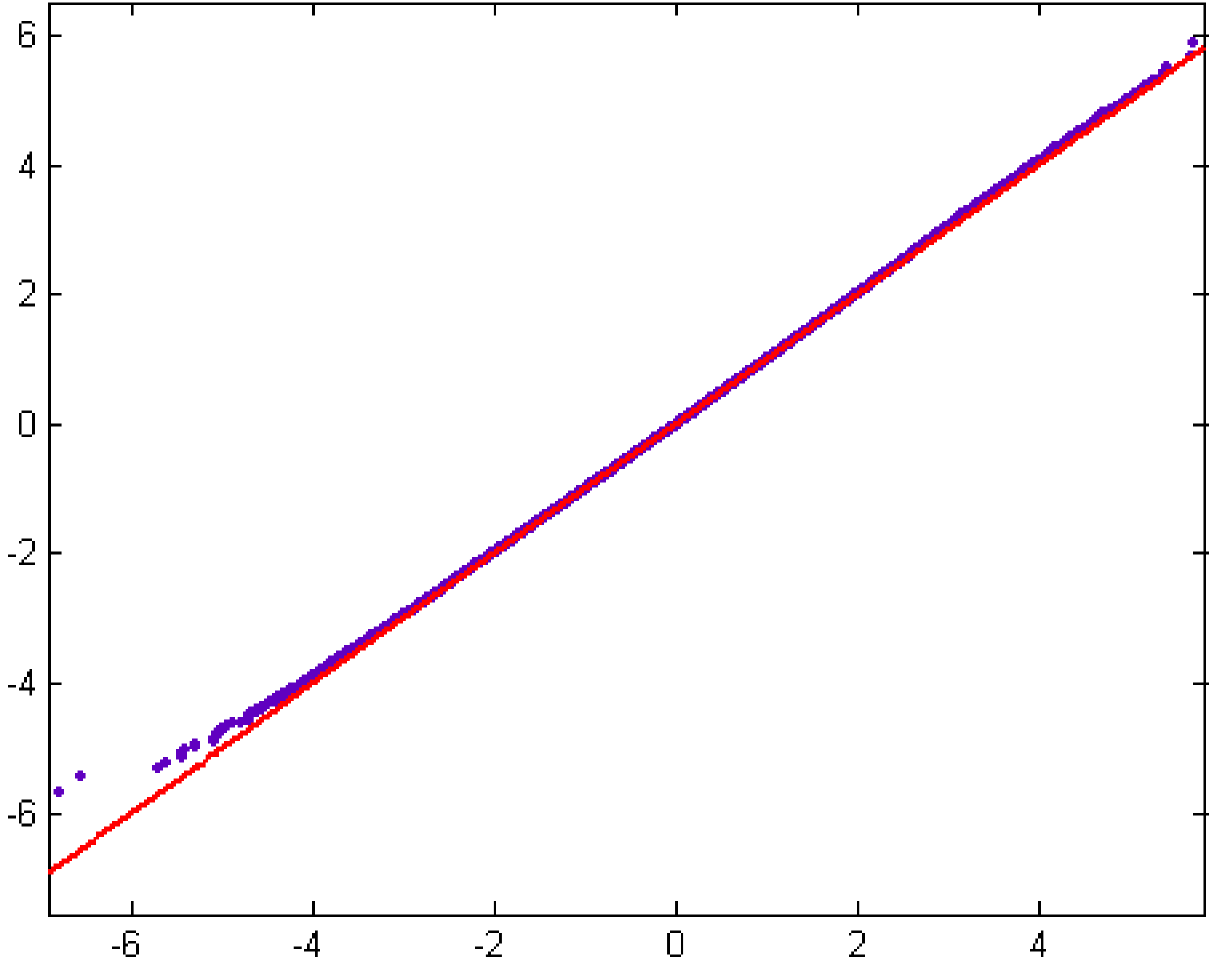


Figure 73 ($n=p^2qrs$, mean=0.0999, standard deviation=1.3869, size=64000)



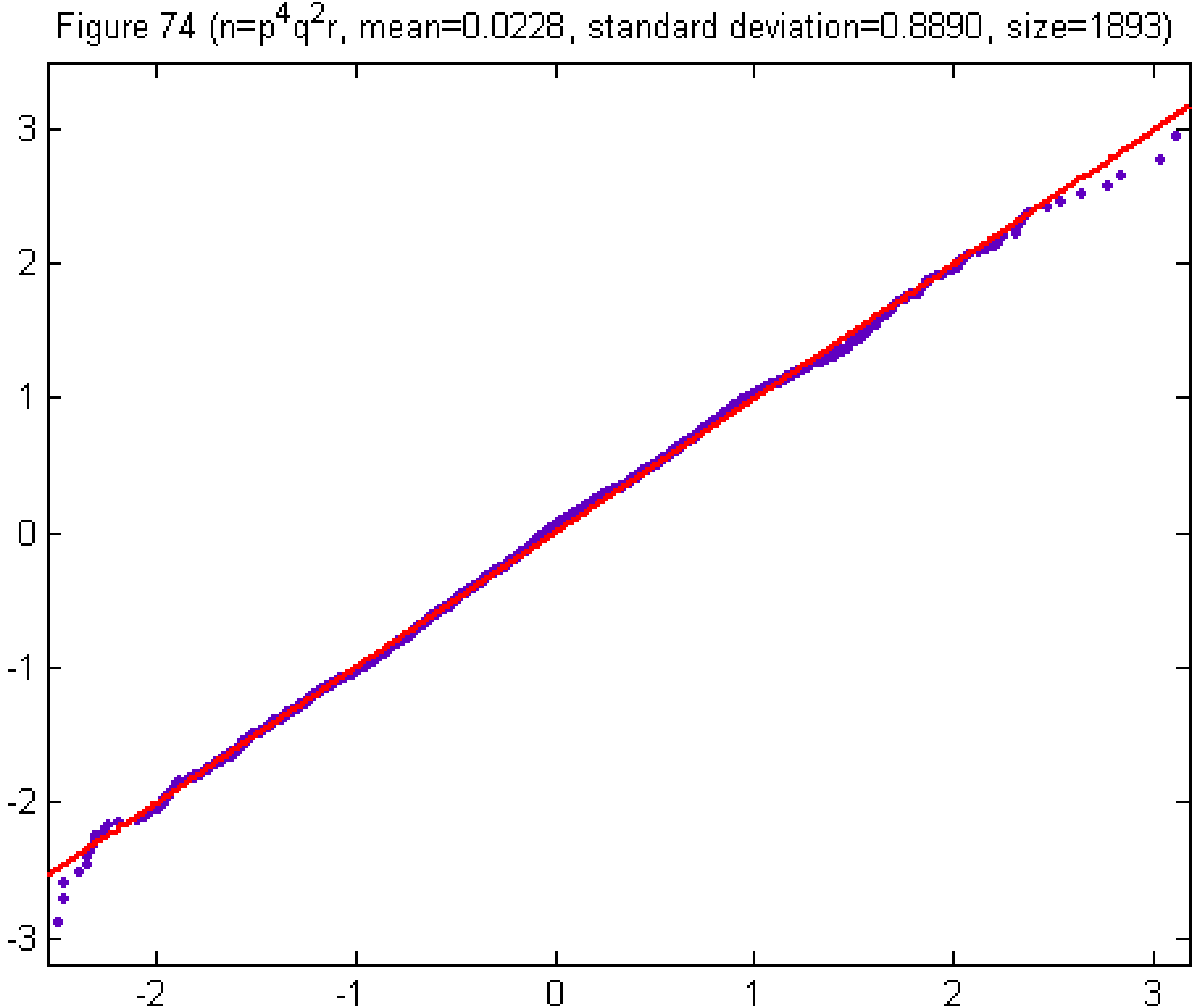


Figure 74 (n=p4q2r, mean=0.0228, standard deviation=0.8890, size=1893)
3
2
1
0
-1
-2
-3
-2
-1
0
1
2
3

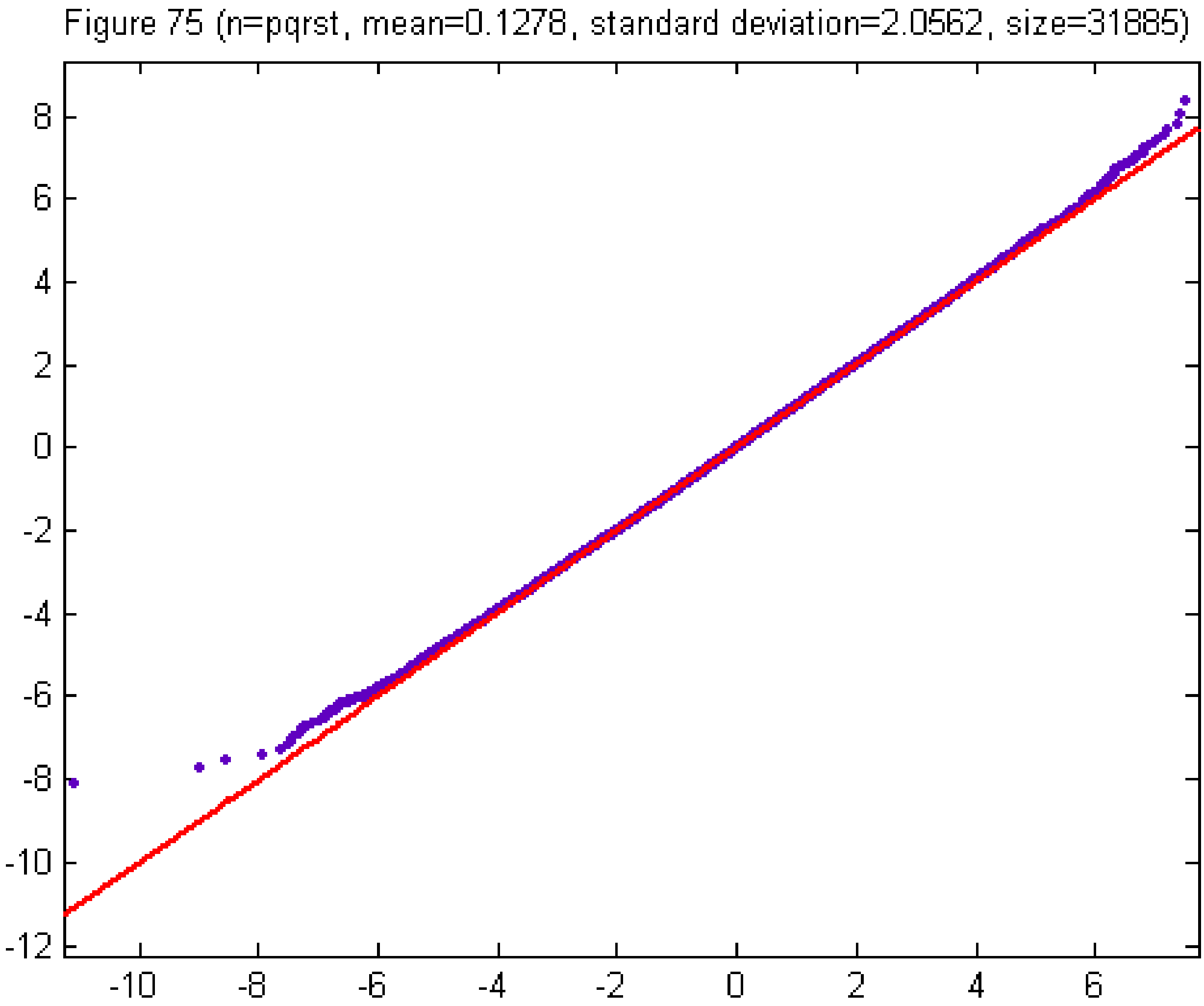


Figure 75 (n=pqrst, mean=0.1278, standard deviation=2.0562, size=31885)
8
6
4
2
0
-2
-4
-6
-8
-10
-12
-10
-8
-6
-4
-2
0
2
4
6

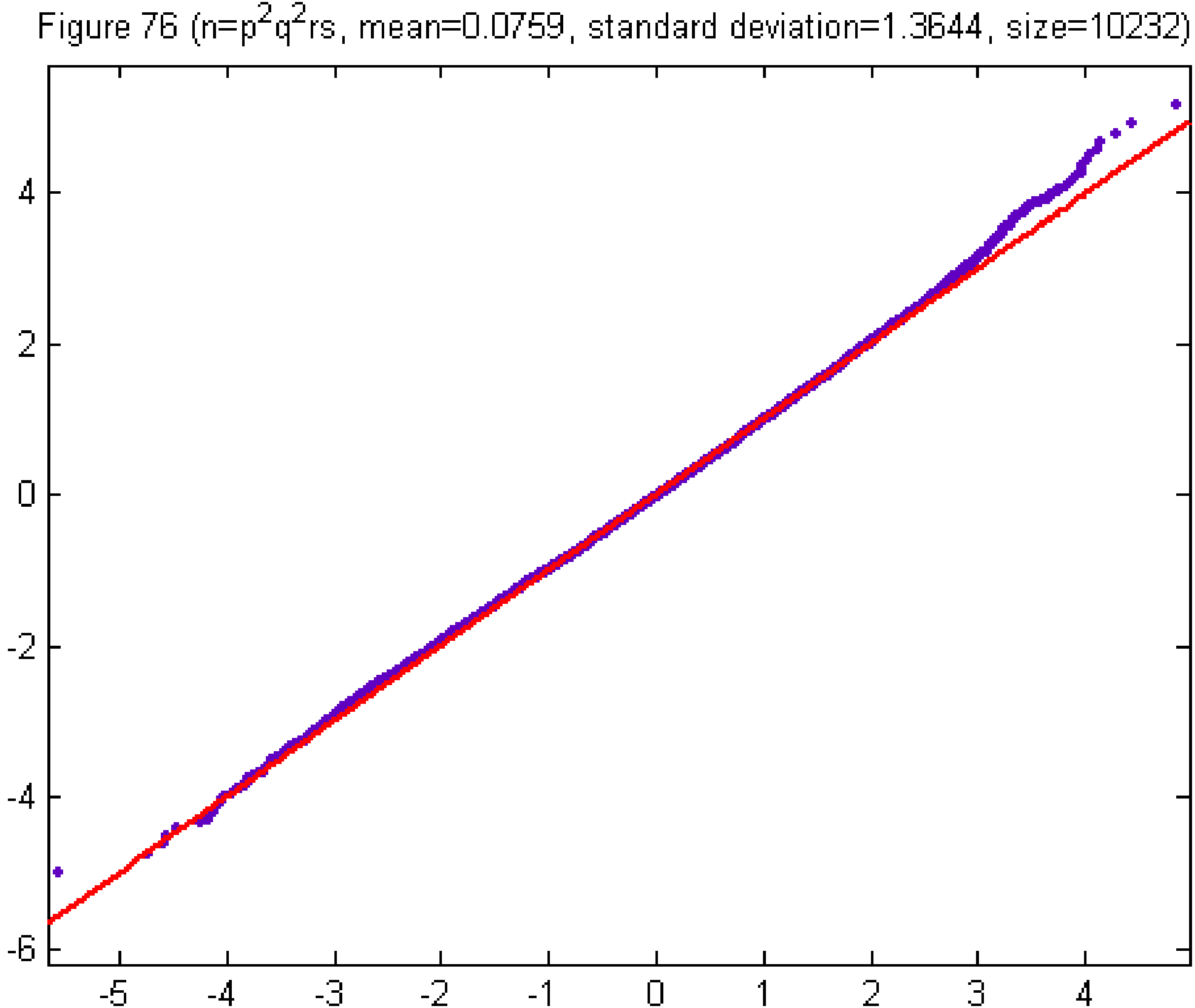


Figure 76 (n=p2q2rs, mean=0.0759, standard deviation=1.3644, size=10232)
4
2
0
-2
-4
-6
-5
-4
-3
-2
-1
0
1
2
3
4

Figure 77 ($n=p^3q^3rs$, mean=0.0784, standard deviation=1.3139, size=1034)

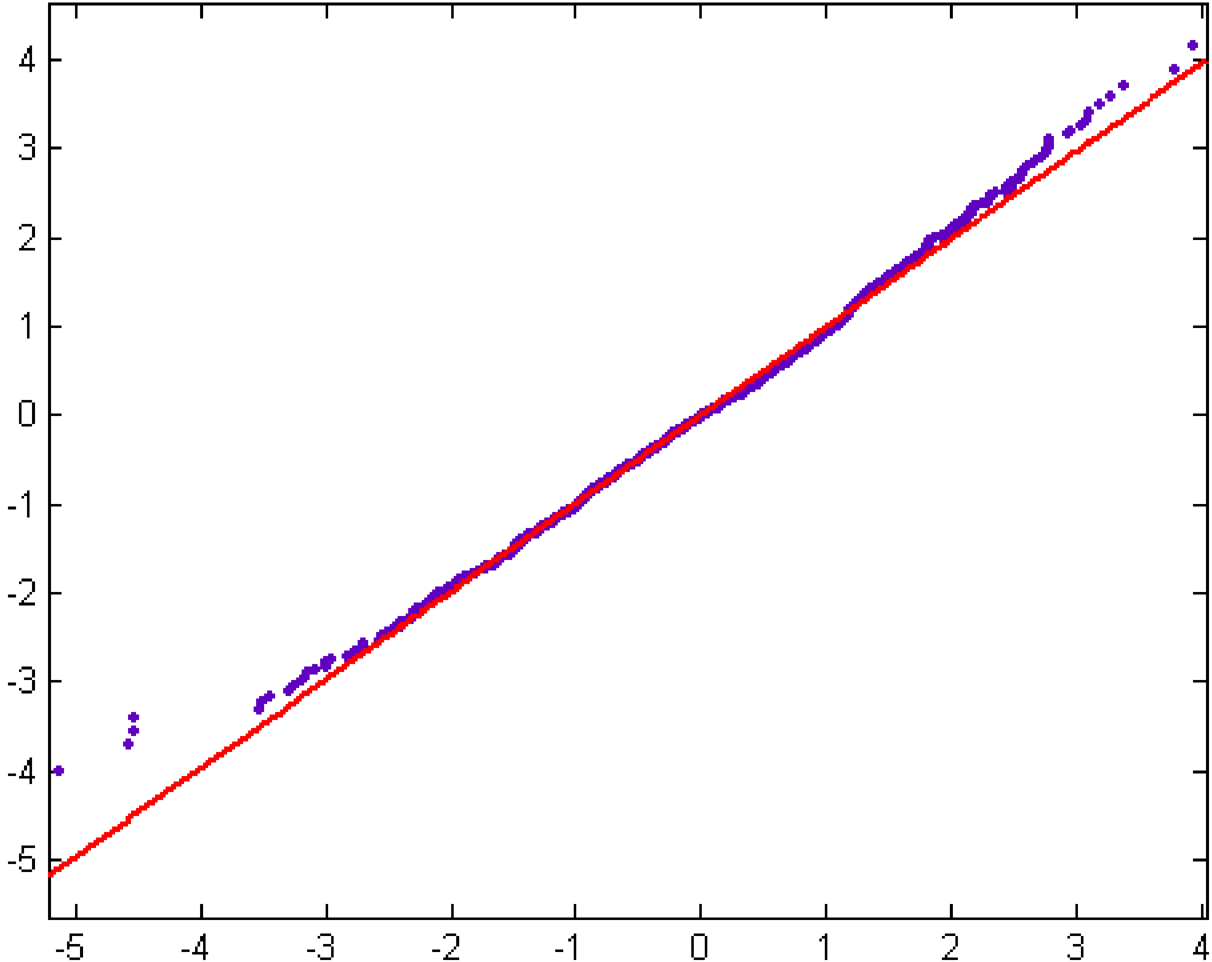


Figure 78 (n=pq, mean=-0.0552, standard deviation=0.6525, size=183836)

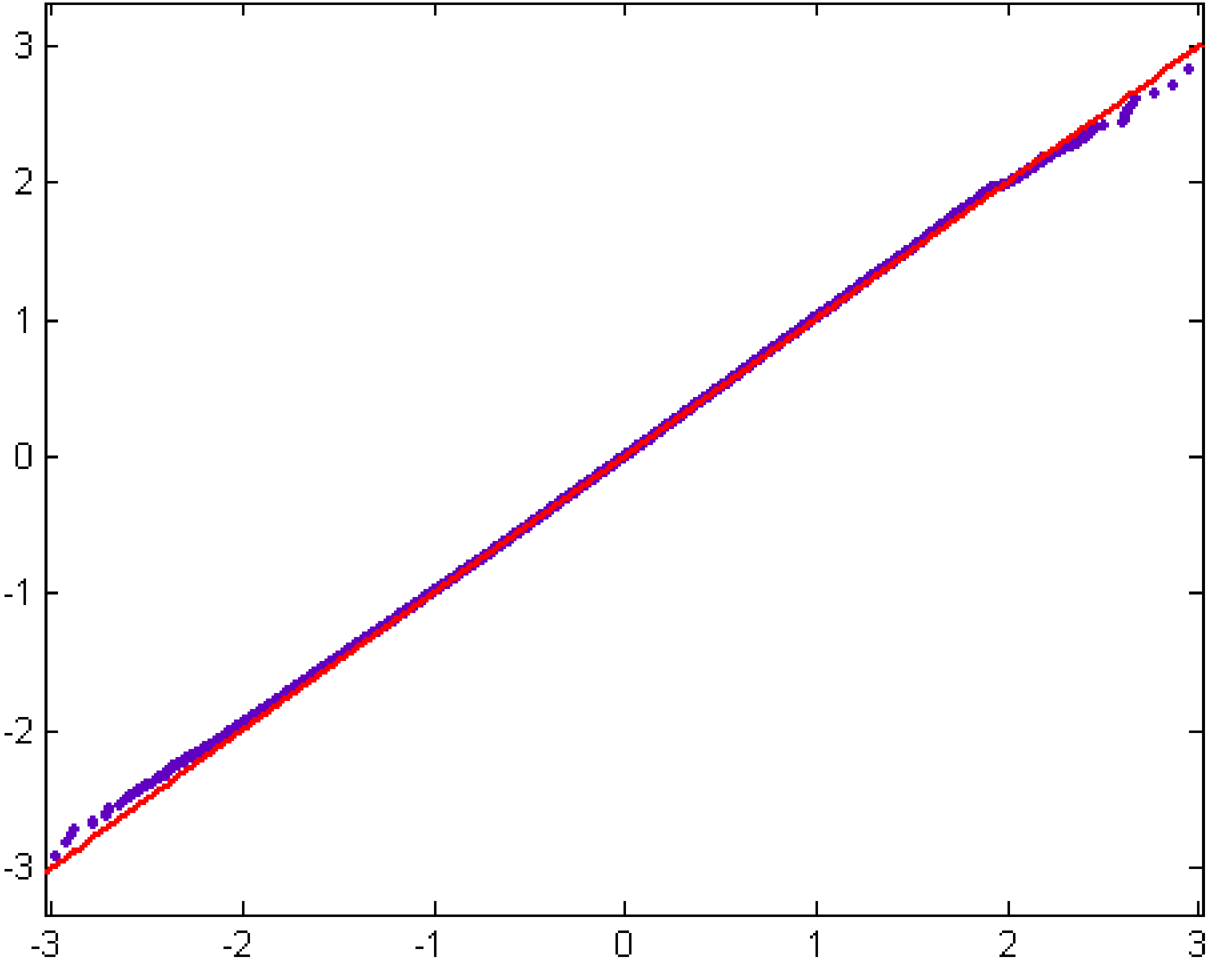


Figure 79 (n=pqr, mean=0.0028, standard deviation=0.9779, size=208653)

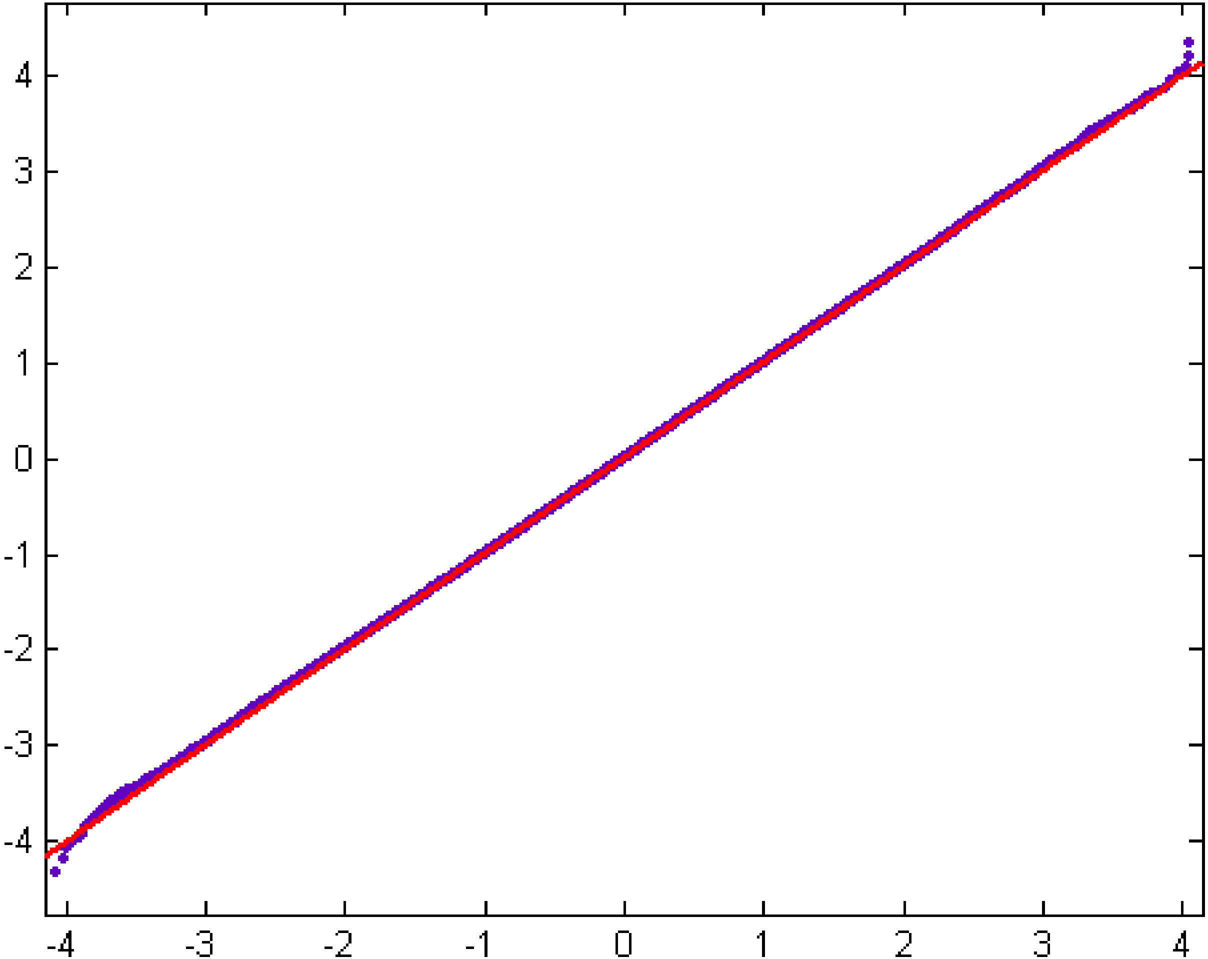
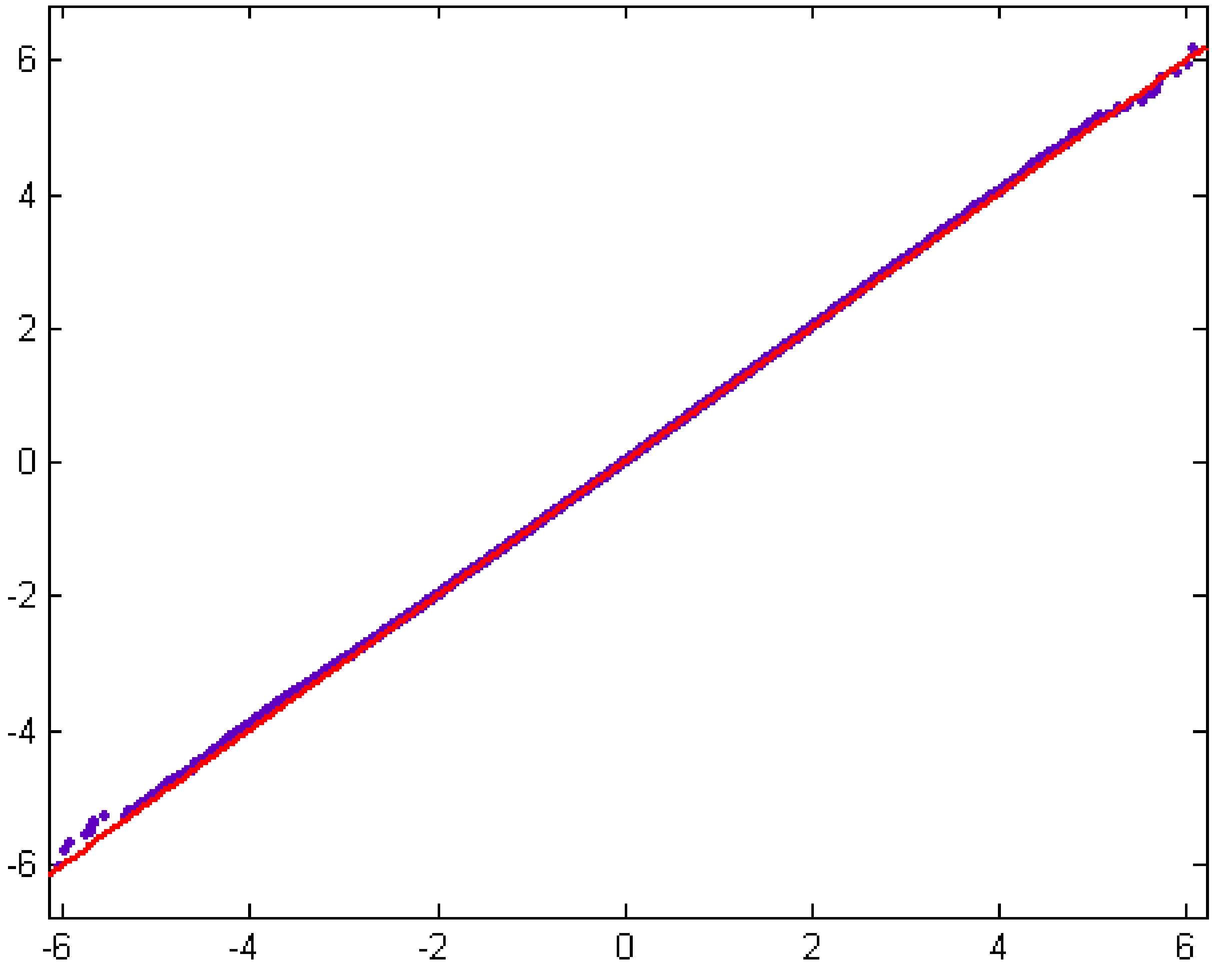


Figure 80 (n=pqrs, mean=0.0697, standard deviation=1.4222, size=115969)



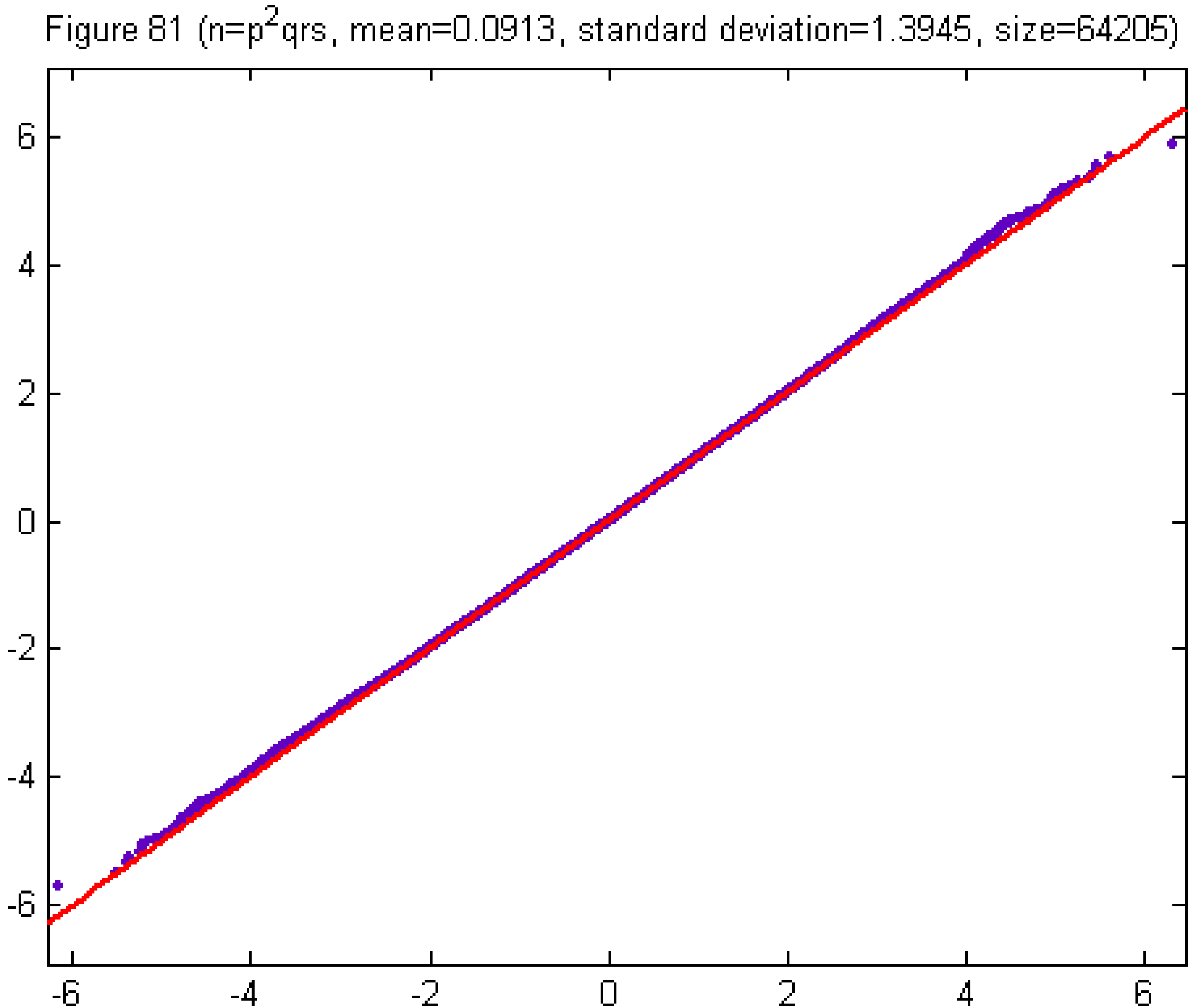


Figure 81 (n=p2qrs, mean=0.0913, standard deviation=1.3945, size=64205)
6
4
2
0
-2
-4
-6
-6
-4
-2
0
2
4
6

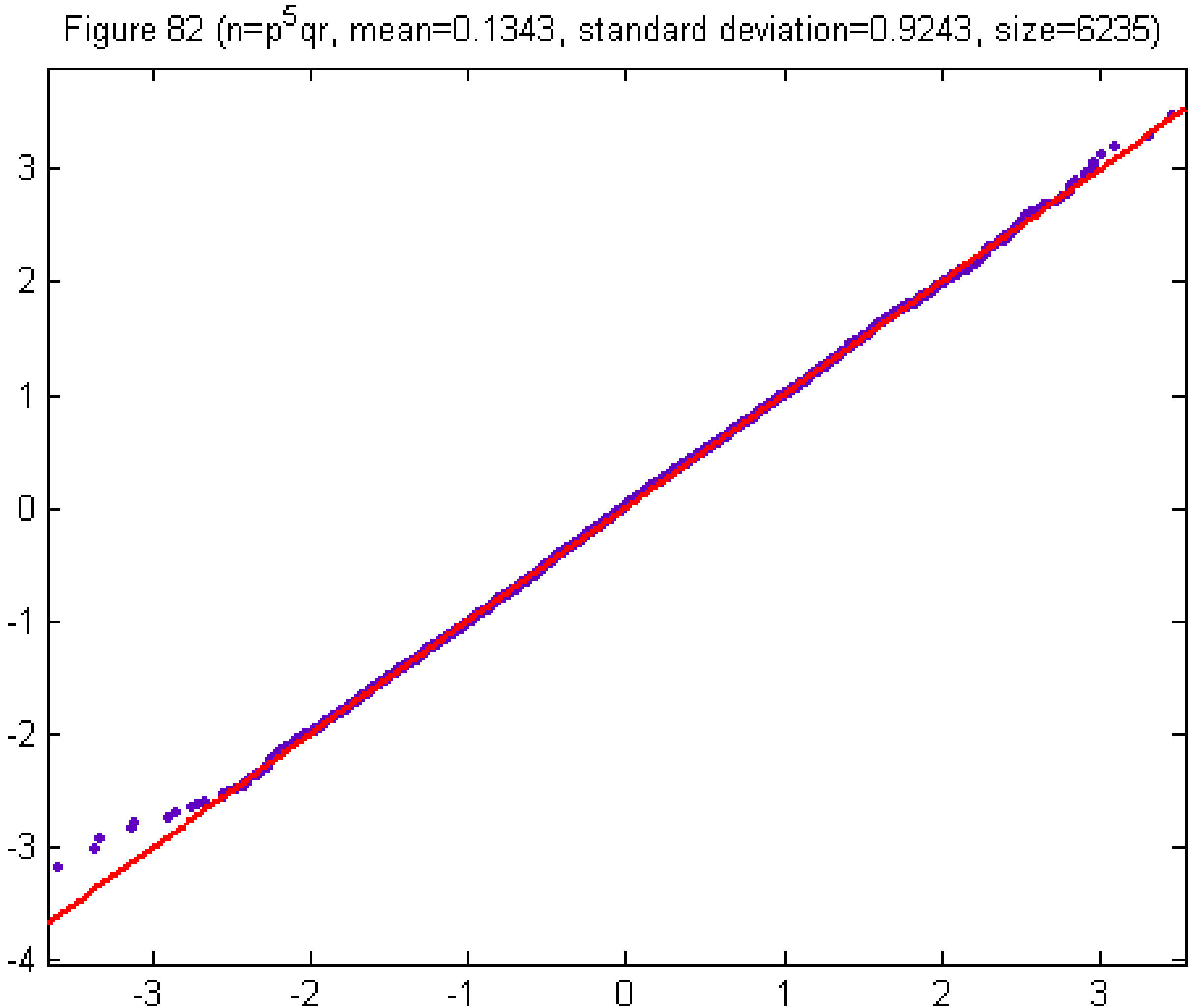


Figure 82 ($n=p^5qr$, mean=0.1343, standard deviation=0.9243, size=6235)
3
2
1
0
-1
-2
-3
-4
-3
-2
-1
0
1
2
3

Figure 83 ($n=p^3qrs$, mean=0.1177, standard deviation=1.3762, size=23148)

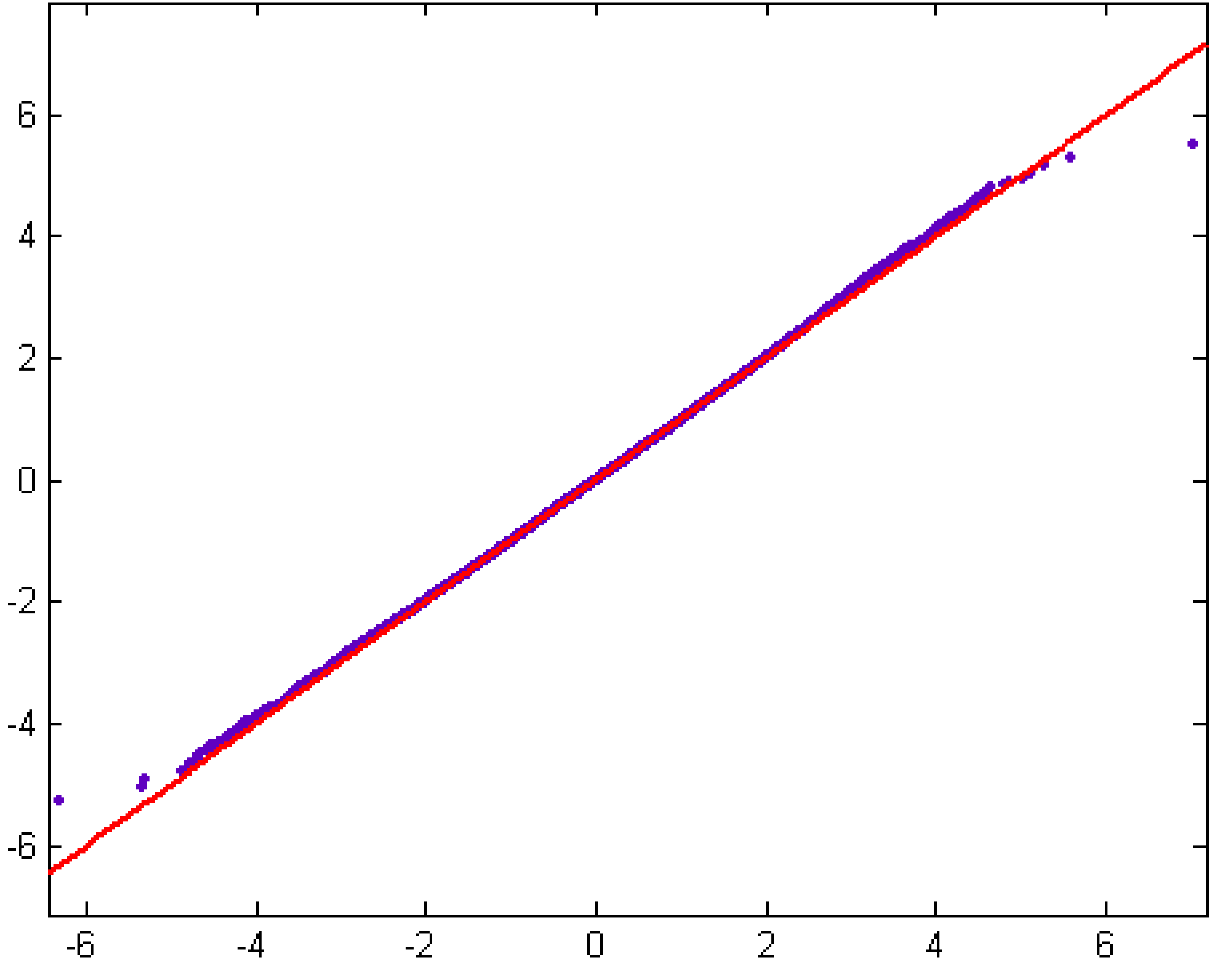
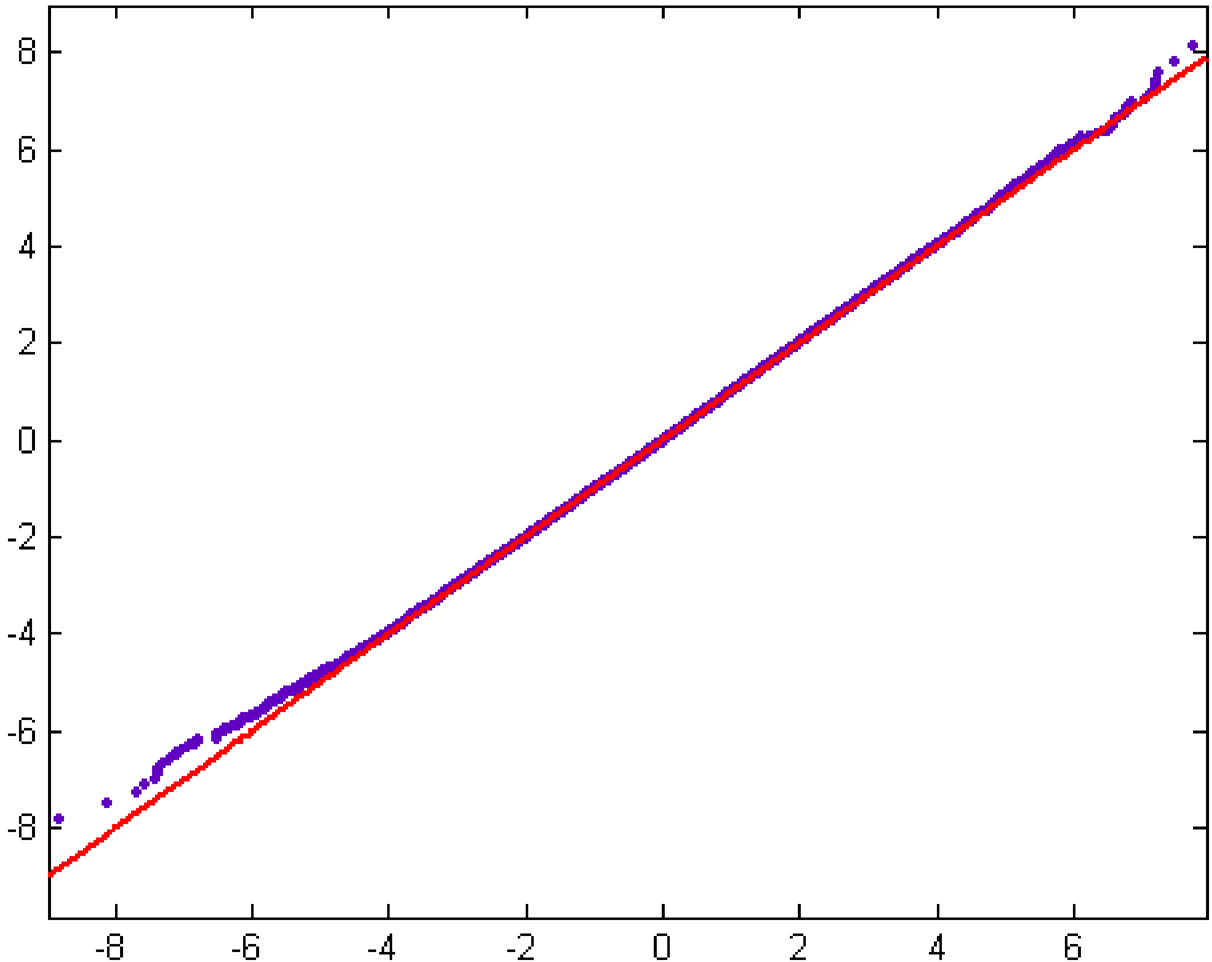


Figure 84 ($n=p^2qrst$, mean=0.1546, standard deviation=2.0359, size=22238)



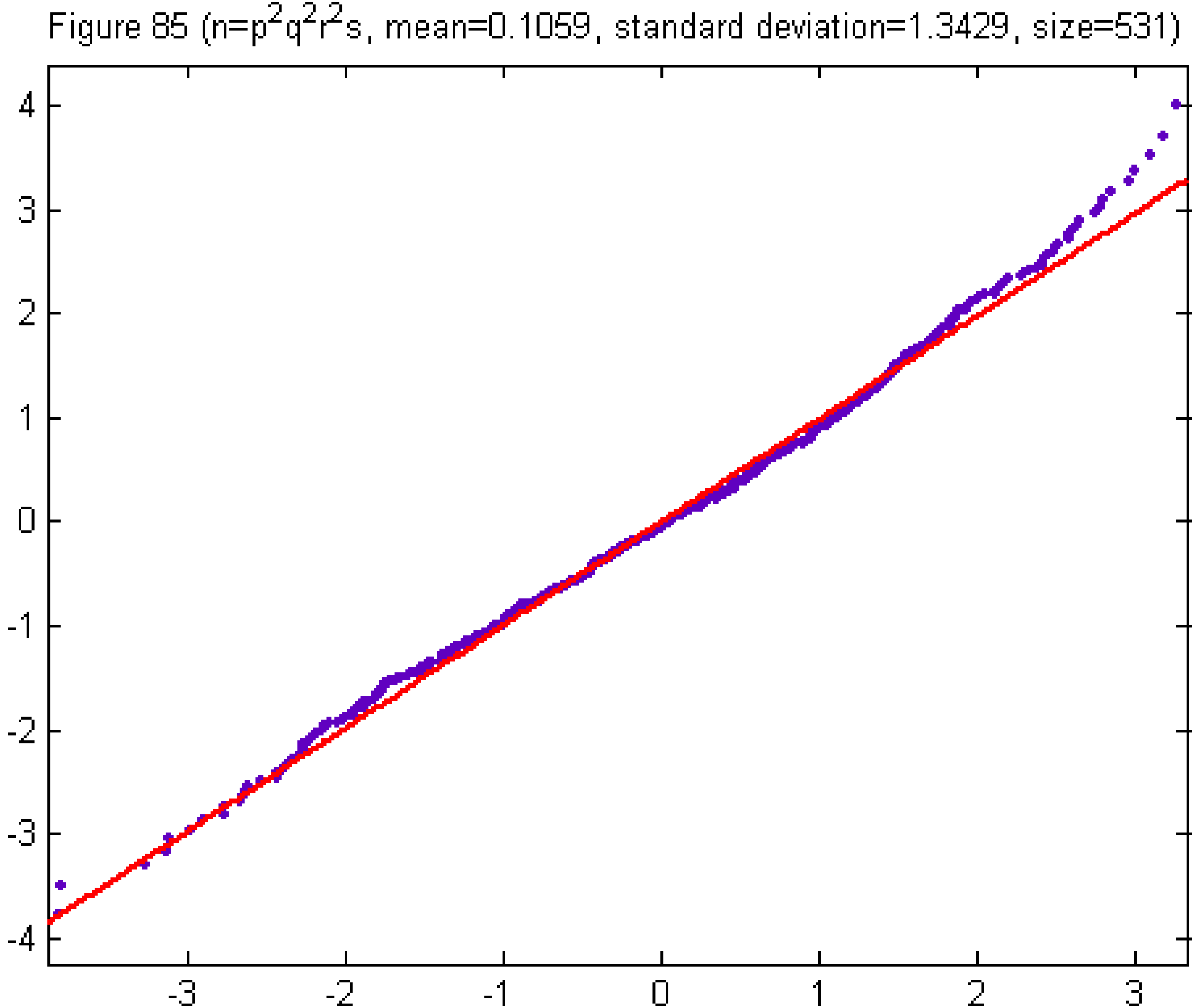


Figure 85 (n=p²q²r²s, mean=0.1059, standard deviation=1.3429, size=531)
4
3
2
1
0
-1
-2
-3
-4
-3
-2
-1
0
1
2
3

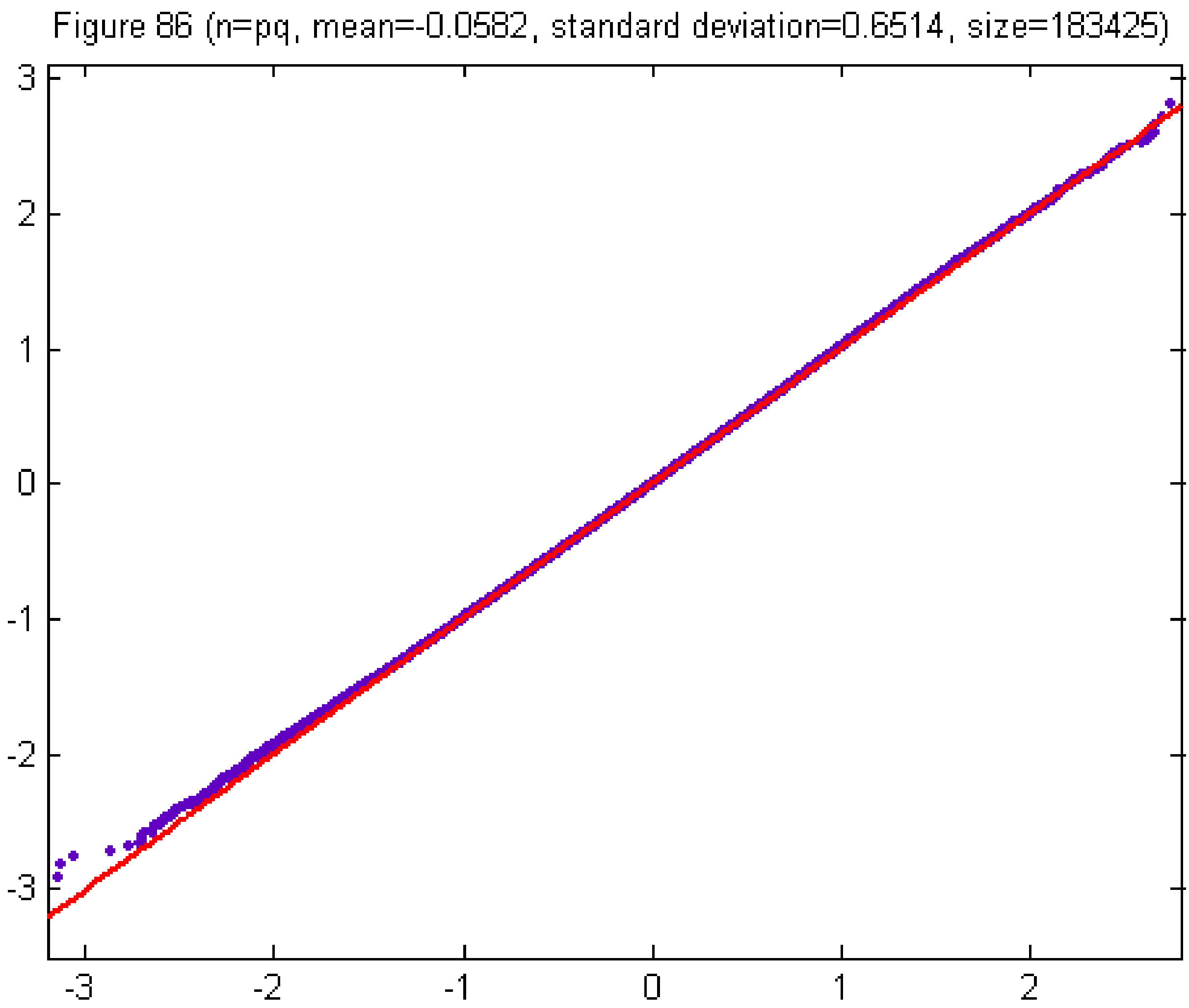


Figure 86 (n=pq, mean=-0.0582, standard deviation=0.6514, size=183425)
3
2
1
0
-1
-2
-3
-3
-2
-1
0
1
2

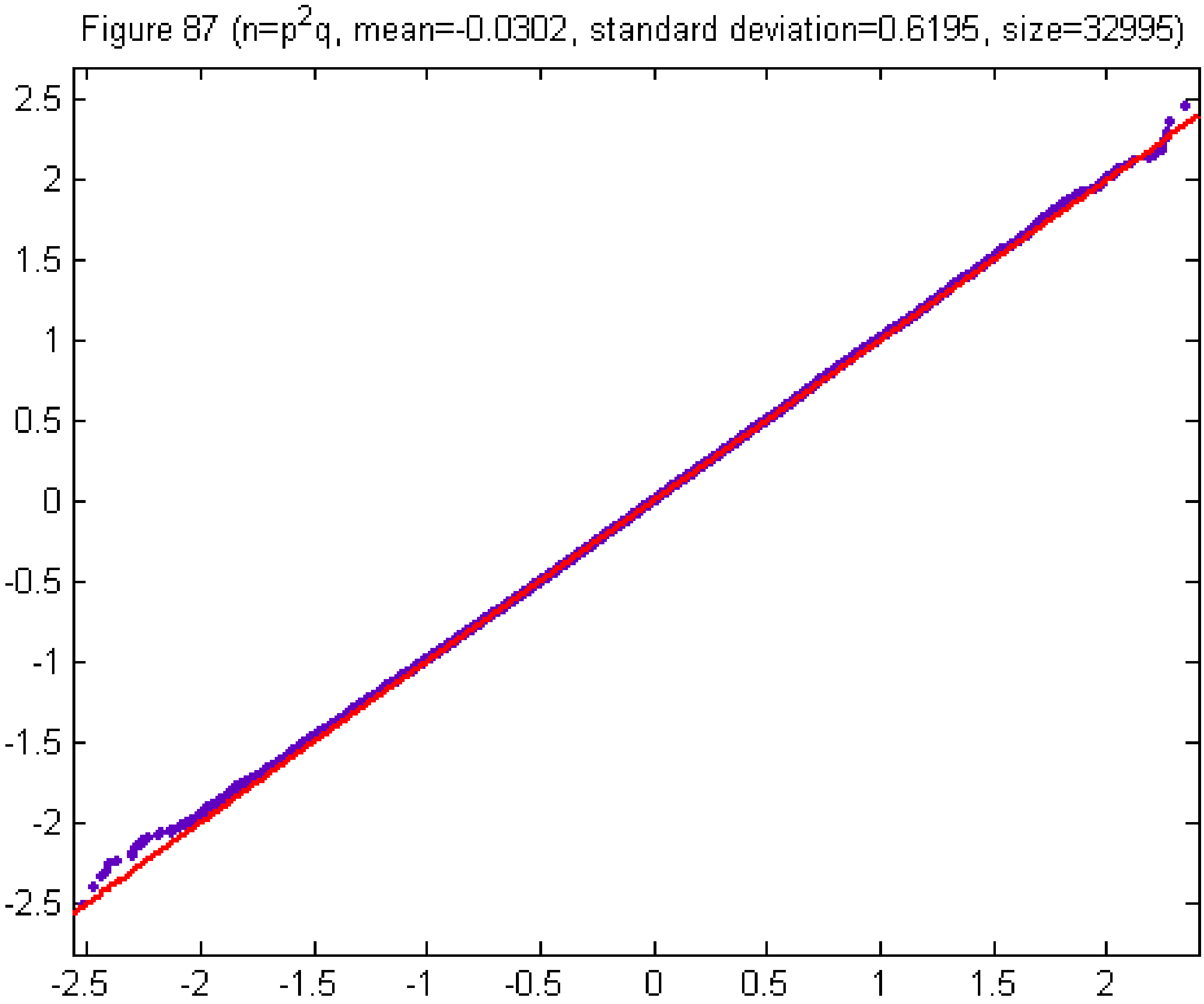


Figure 87 (n=p2q, mean=-0.0302, standard deviation=0.6195, size=32995)
2.5
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
-2.5
-2.5
-2
-1.5
-1
-0.5
0
0.5
1
1.5
2

Figure 88 ($n=p^5q$, mean=0.0886, standard deviation=0.5978, size=2881)

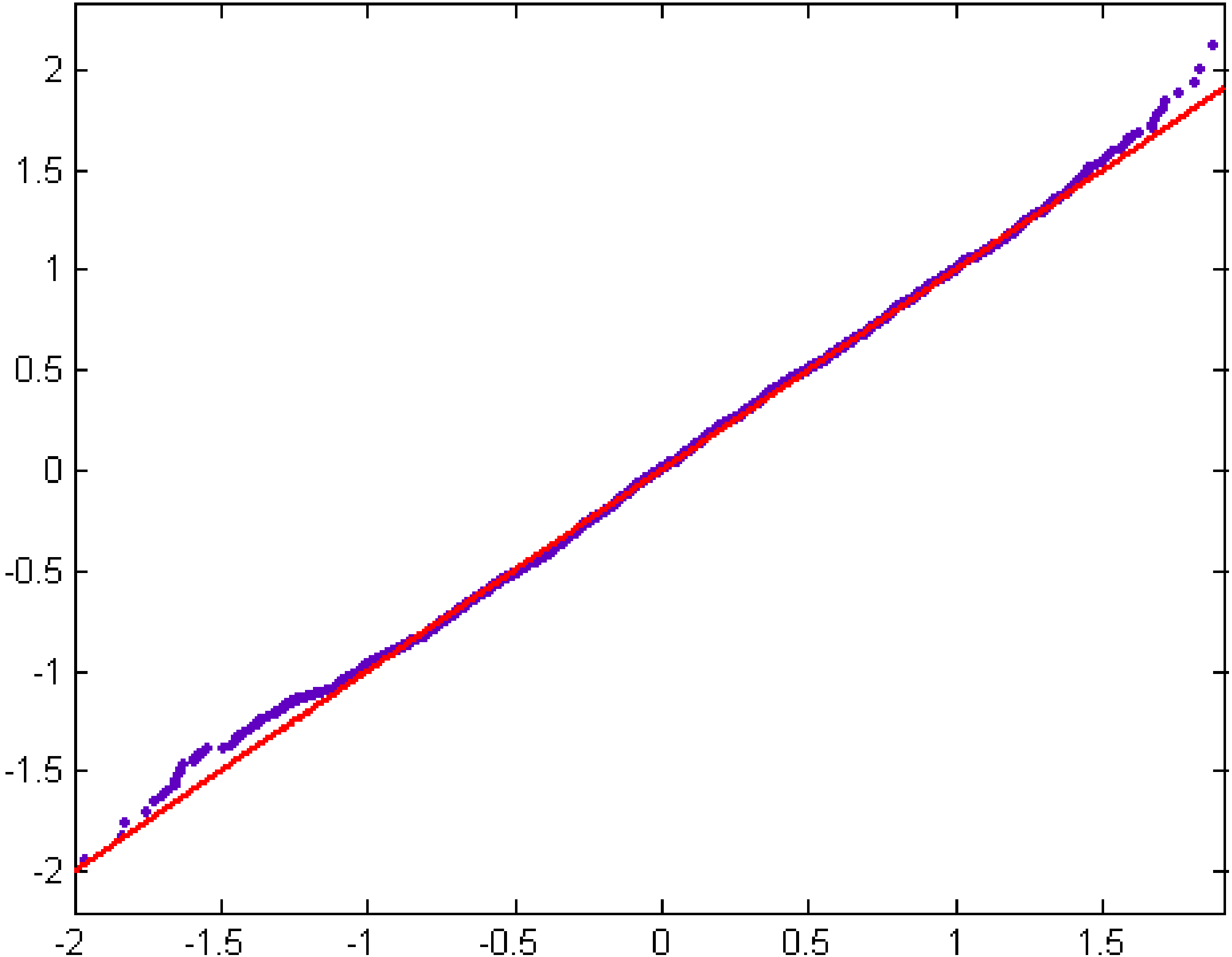


Figure 89 (n=pqrs, mean=0.0754, standard deviation=1.4245, size=116952)

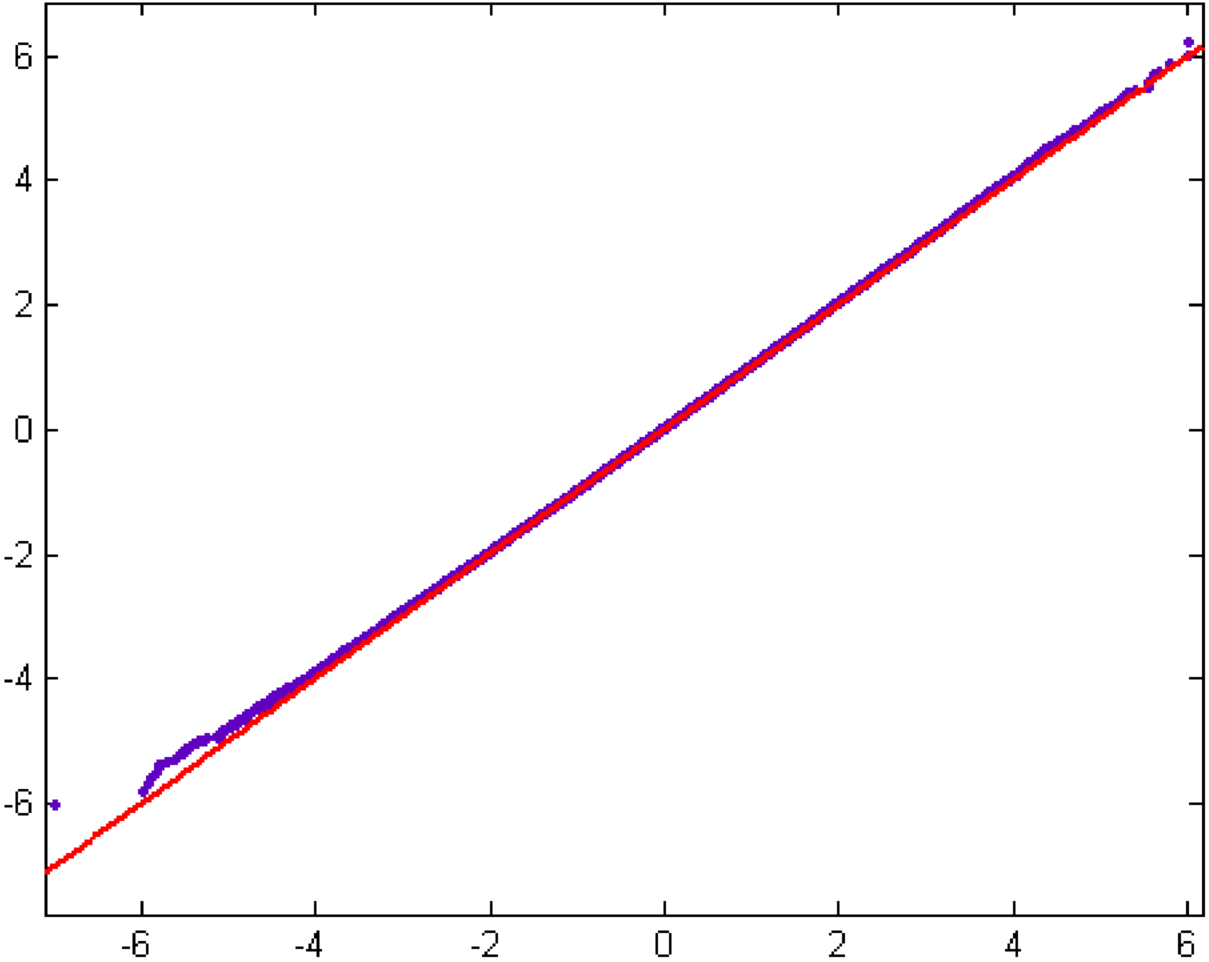


Figure 90 ($n=p^2q^2r$, mean=-0.0085, standard deviaiton=0.9165, size=6062)

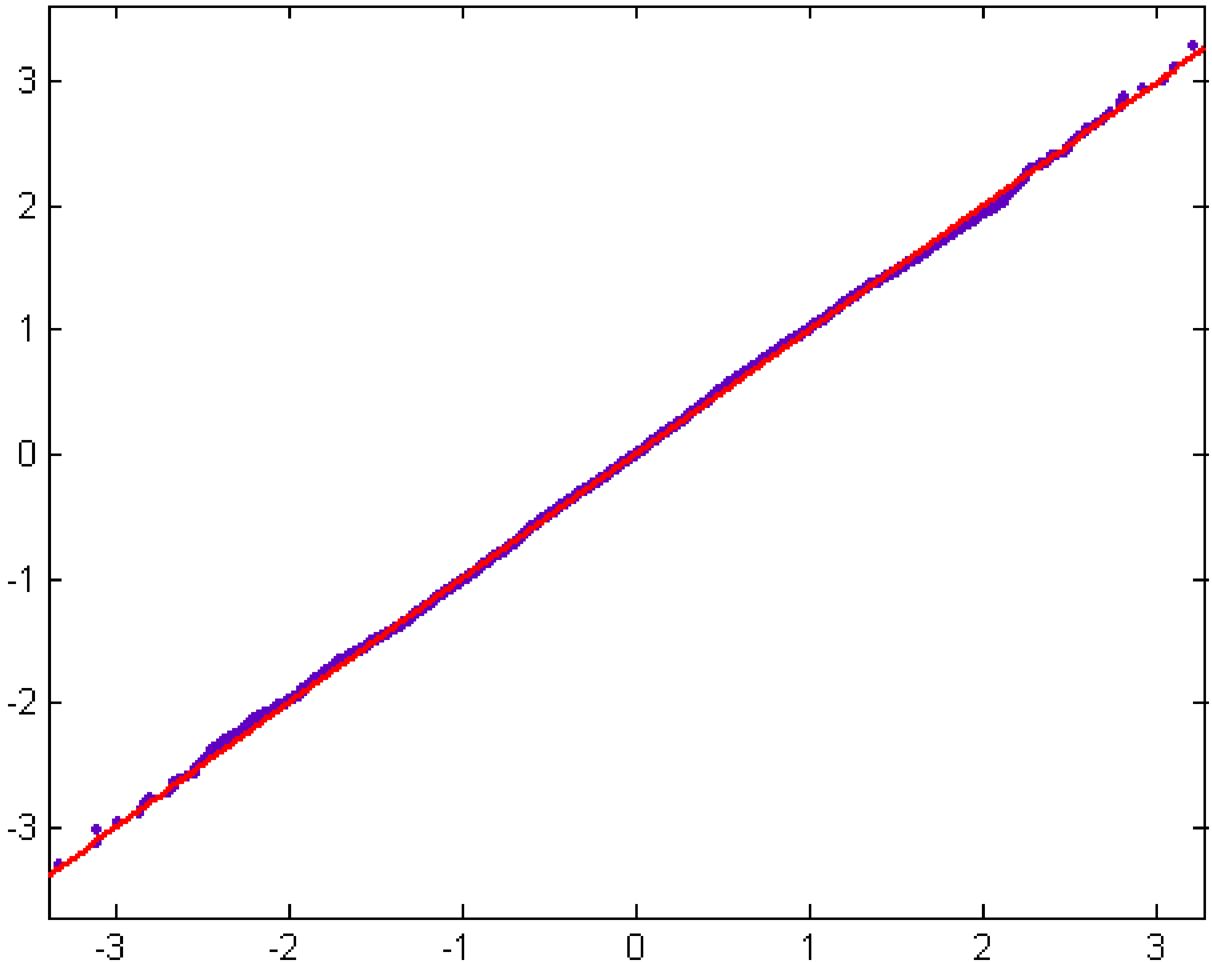
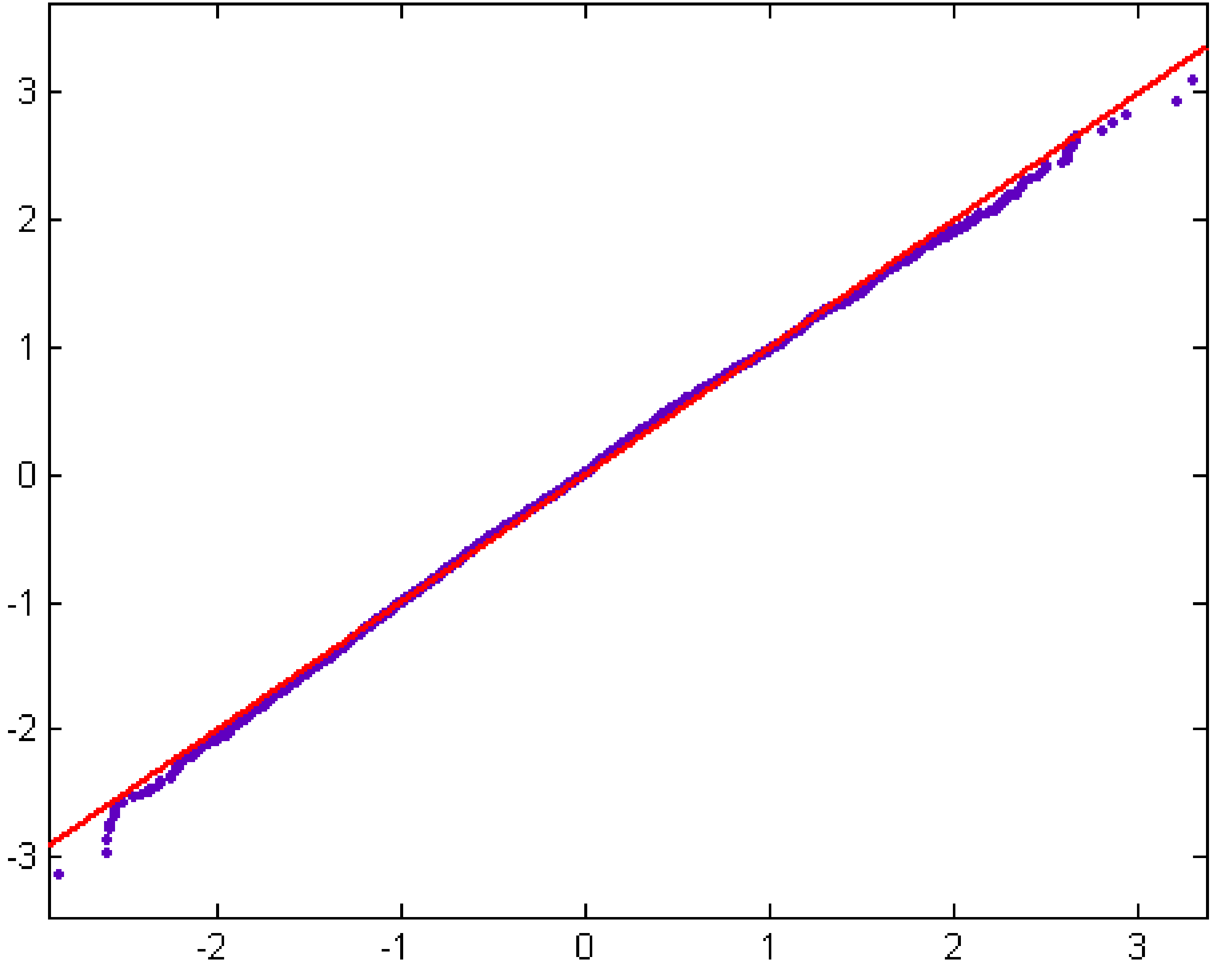


Figure 91 ($n=p^3q^2r$, mean=-0.0198, standard deviation=0.8917, size=4271)



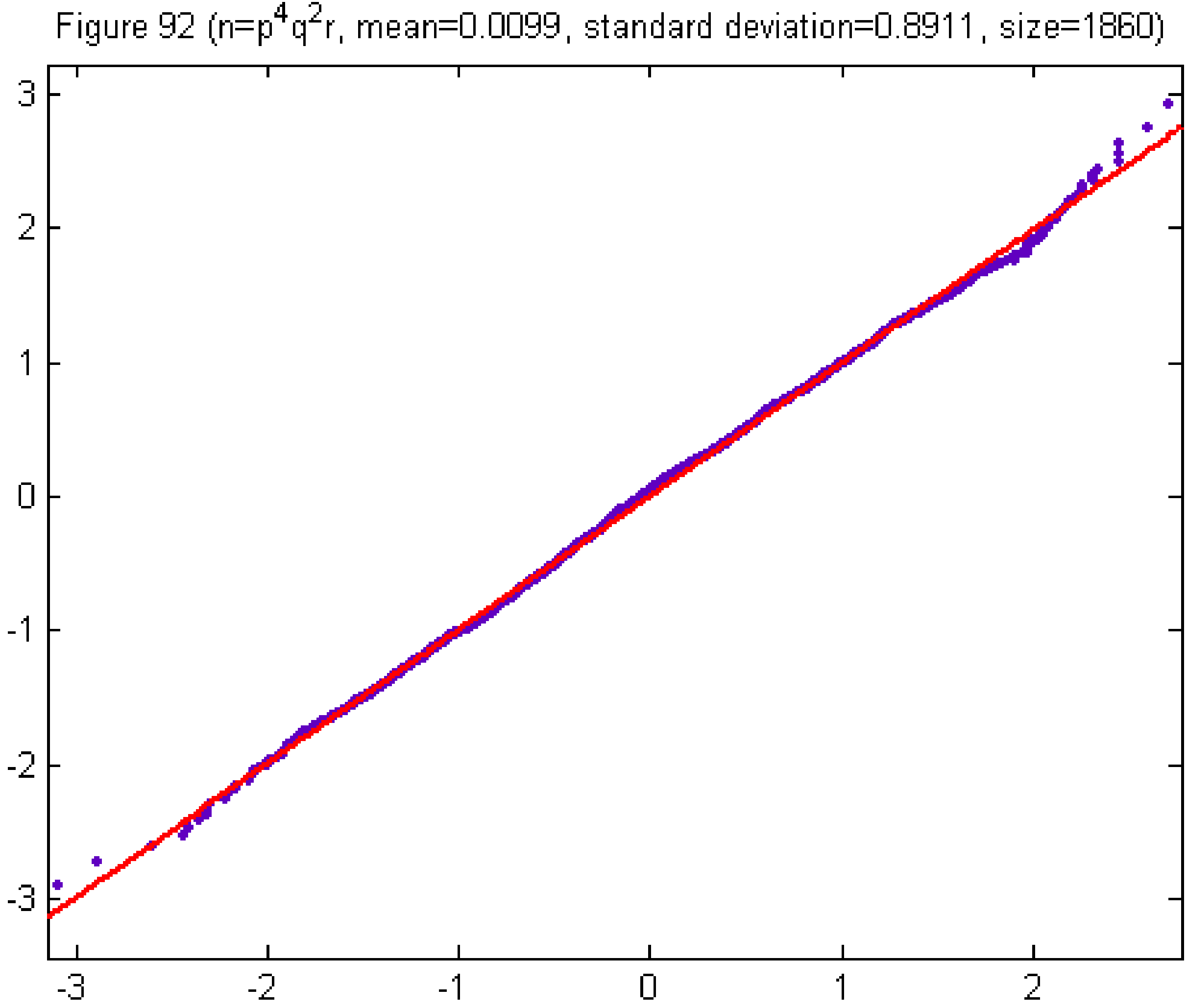


Figure 92 (n=p4q2r, mean=0.0099, standard deviation=0.8911, size=1860)
3
2
1
0
-1
-2
-3
-3
-2
-1
0
1
2

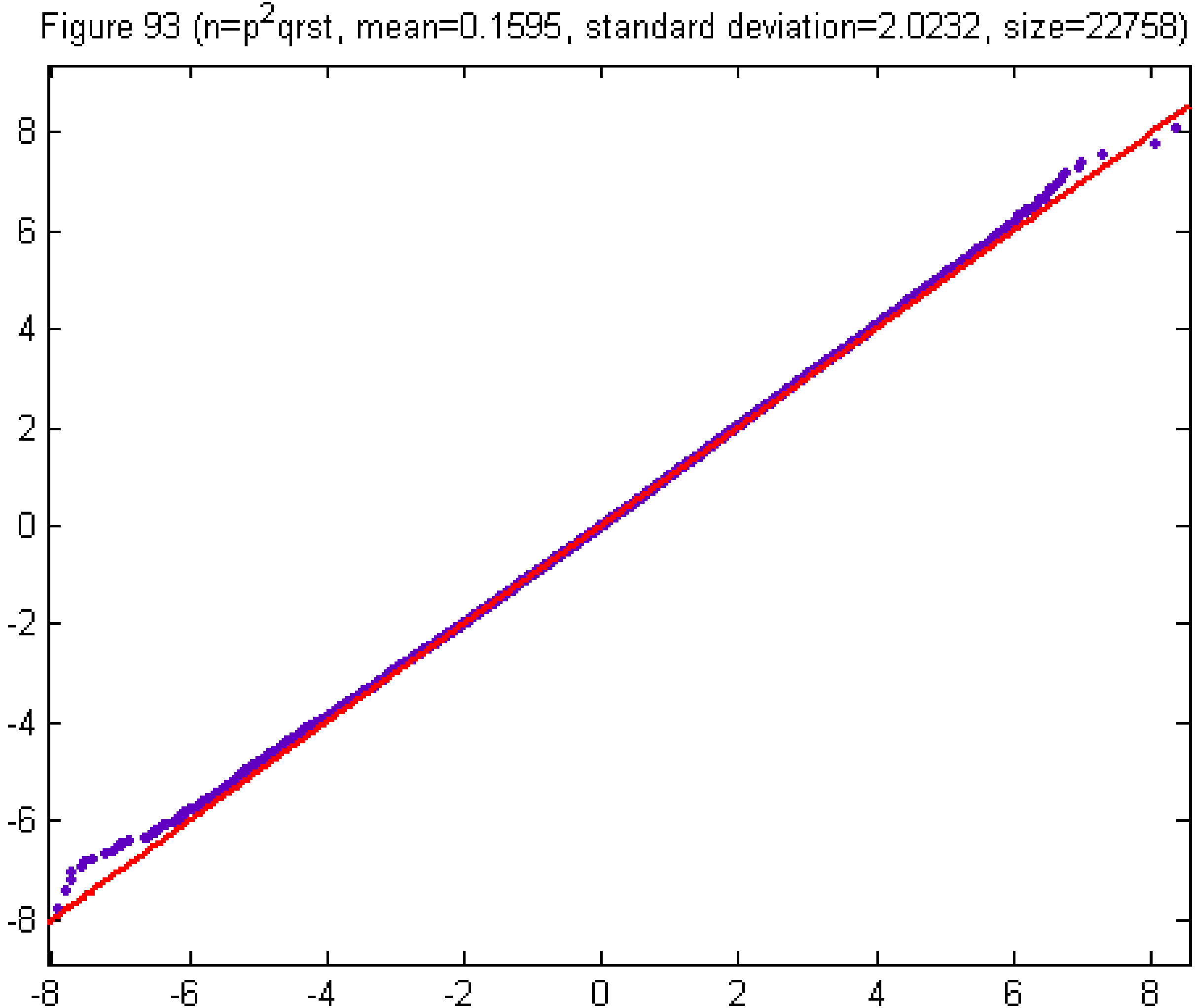


Figure 93 (n=p²qrst, mean=0.1595, standard deviation=2.0232, size=22758)
8
6
4
2
0
-2
-4
-6
-8
-8
-6
-4
-2
0
2
4
6
8

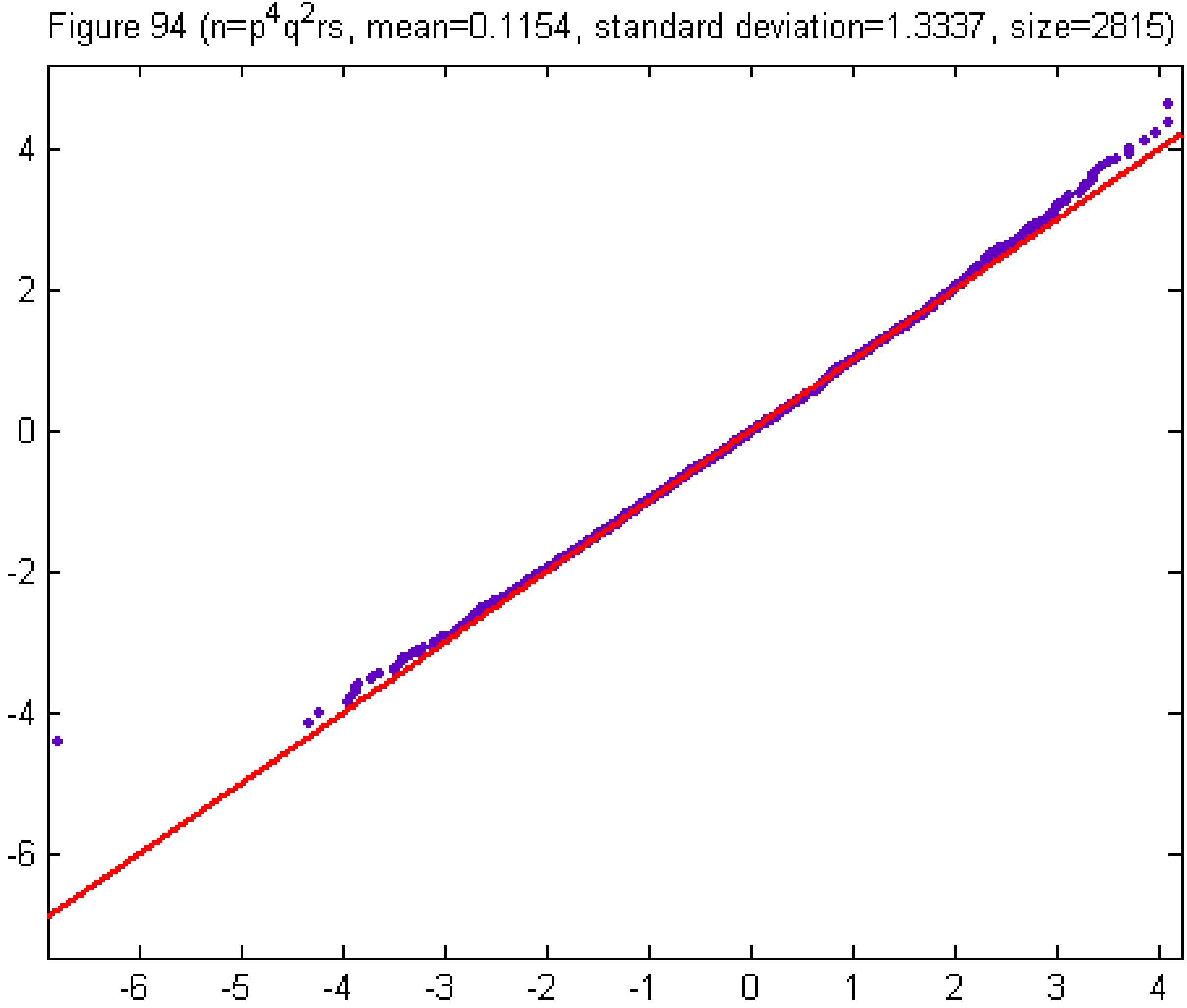


Figure 94 (n=p4q2rs, mean=0.1154, standard deviation=1.3337, size=2815)
4
2
0
-2
-4
-6
-6
-5
-4
-3
-2
-1
0
1
2
3
4

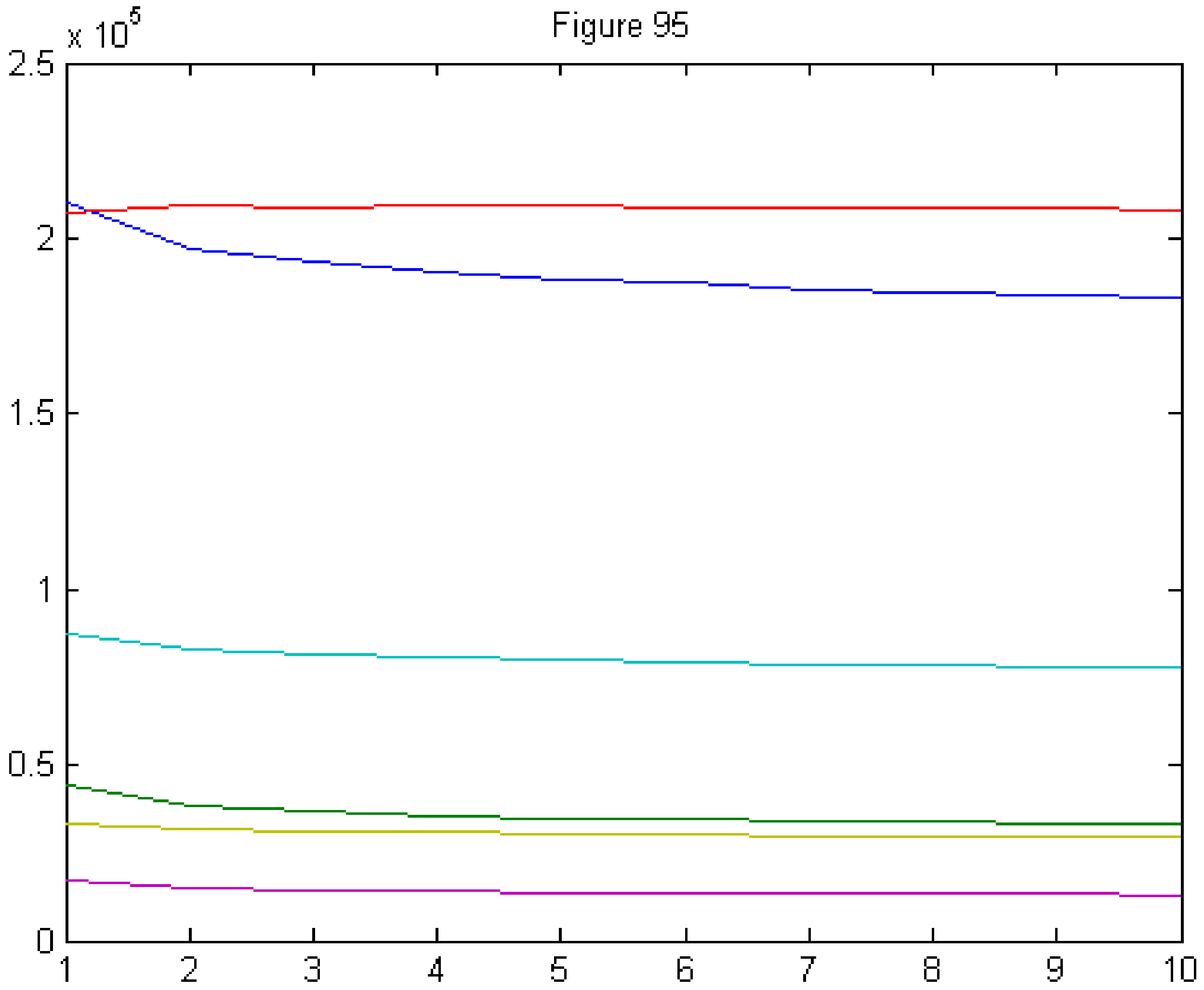


Figure 95
x 10^5
2.5
2
1.5
1
0.5
0
1
2
3
4
5
6
7
8
9
10

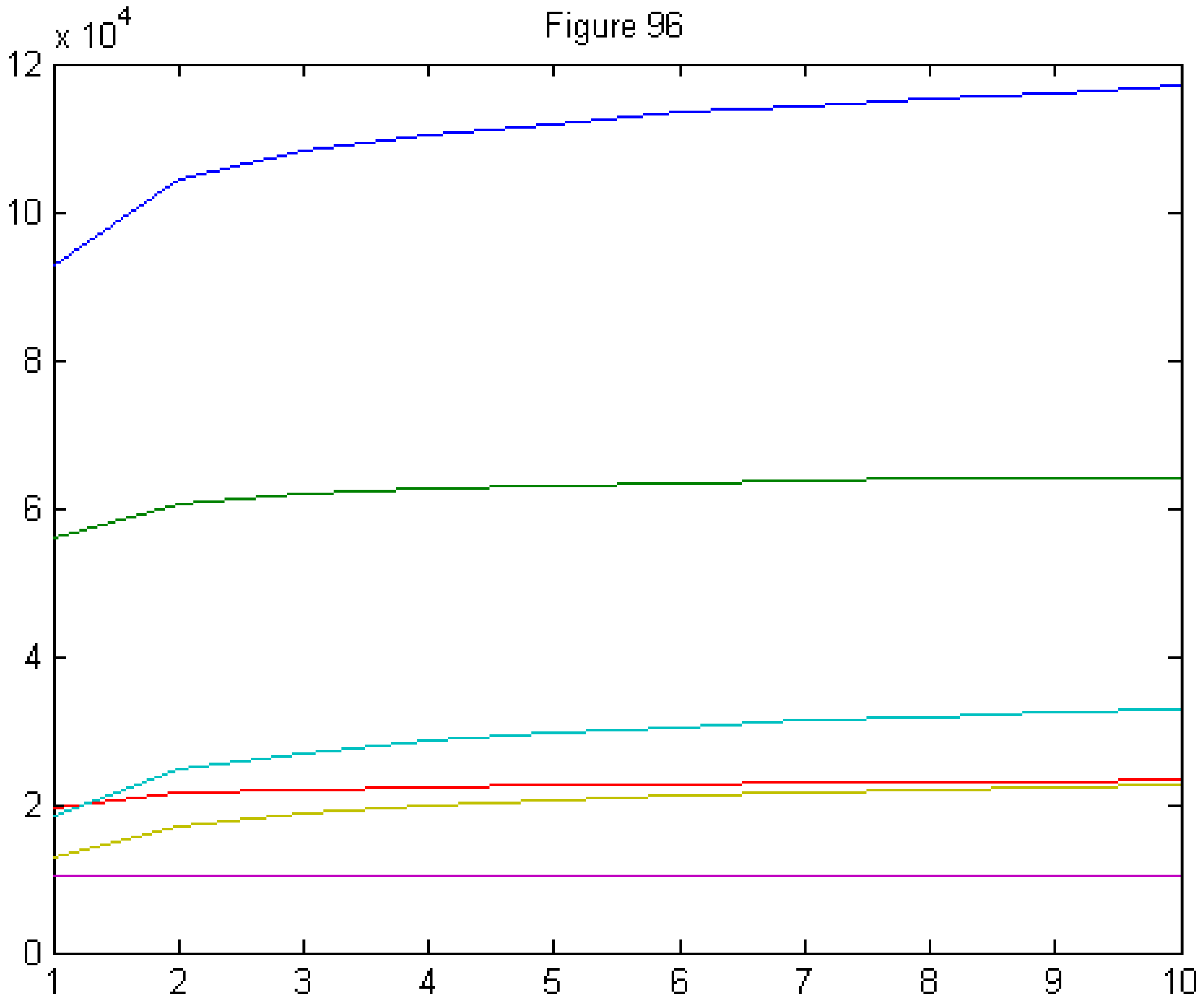


Figure 96
x 10^4
12
10
8
6
4
2
0
1
2
3
4
5
6
7
8
9
10

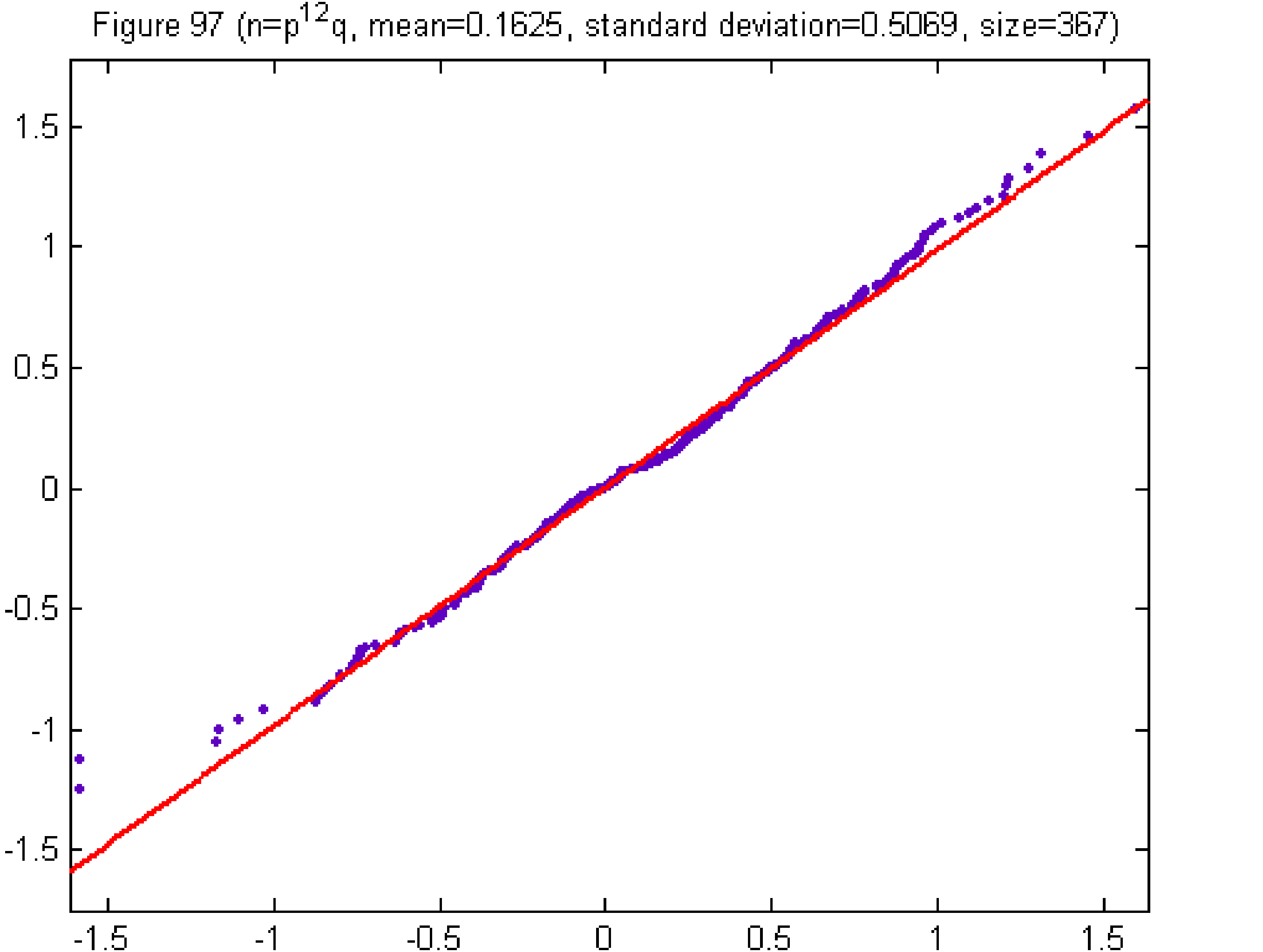


Figure 97 (n=p^12q, mean=0.1625, standard deviation=0.5069, size=367)
1.5
1
0.5
0
-0.5
-1
-1.5
-1.5
-1
-0.5
0
0.5
1
1.5

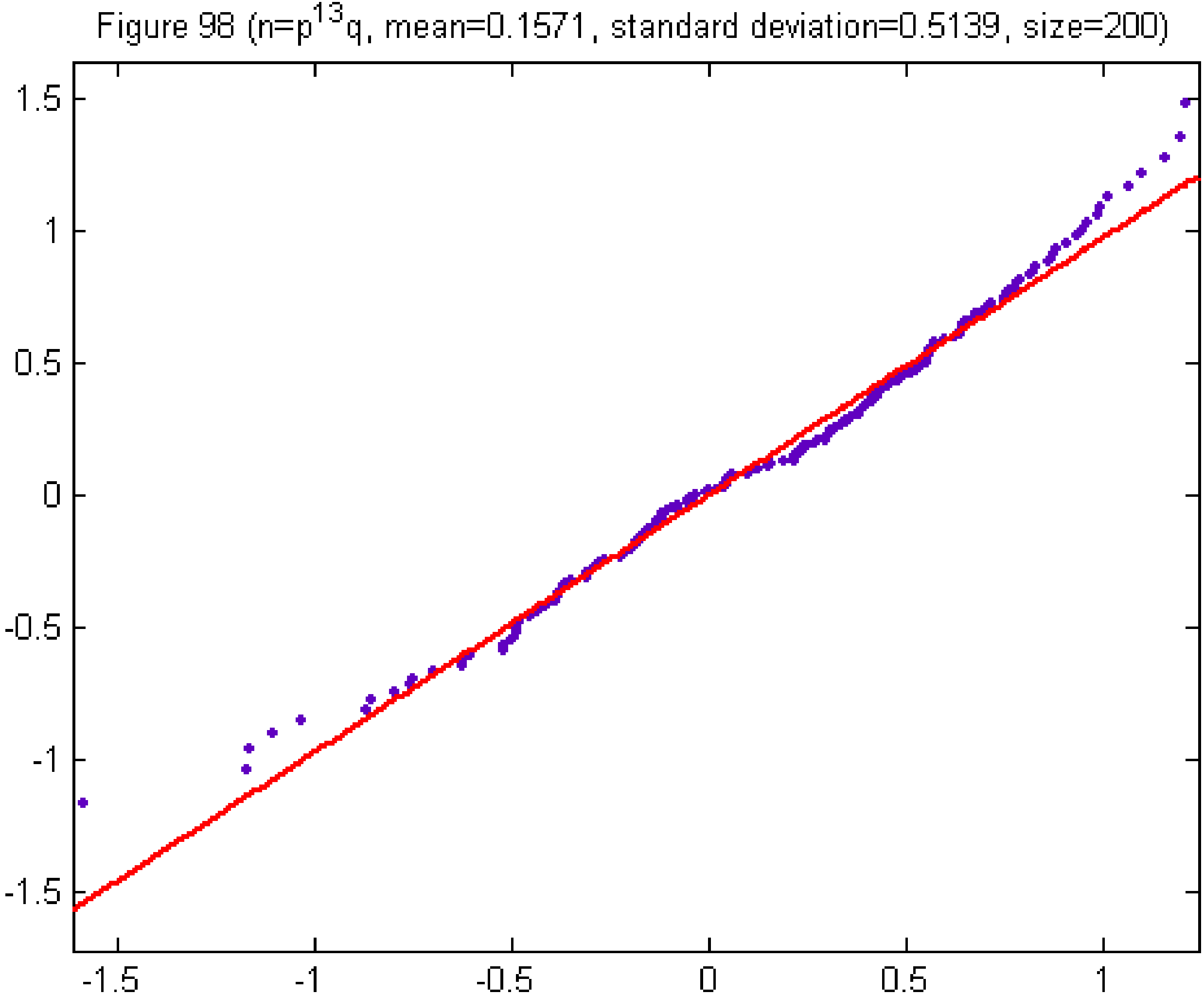
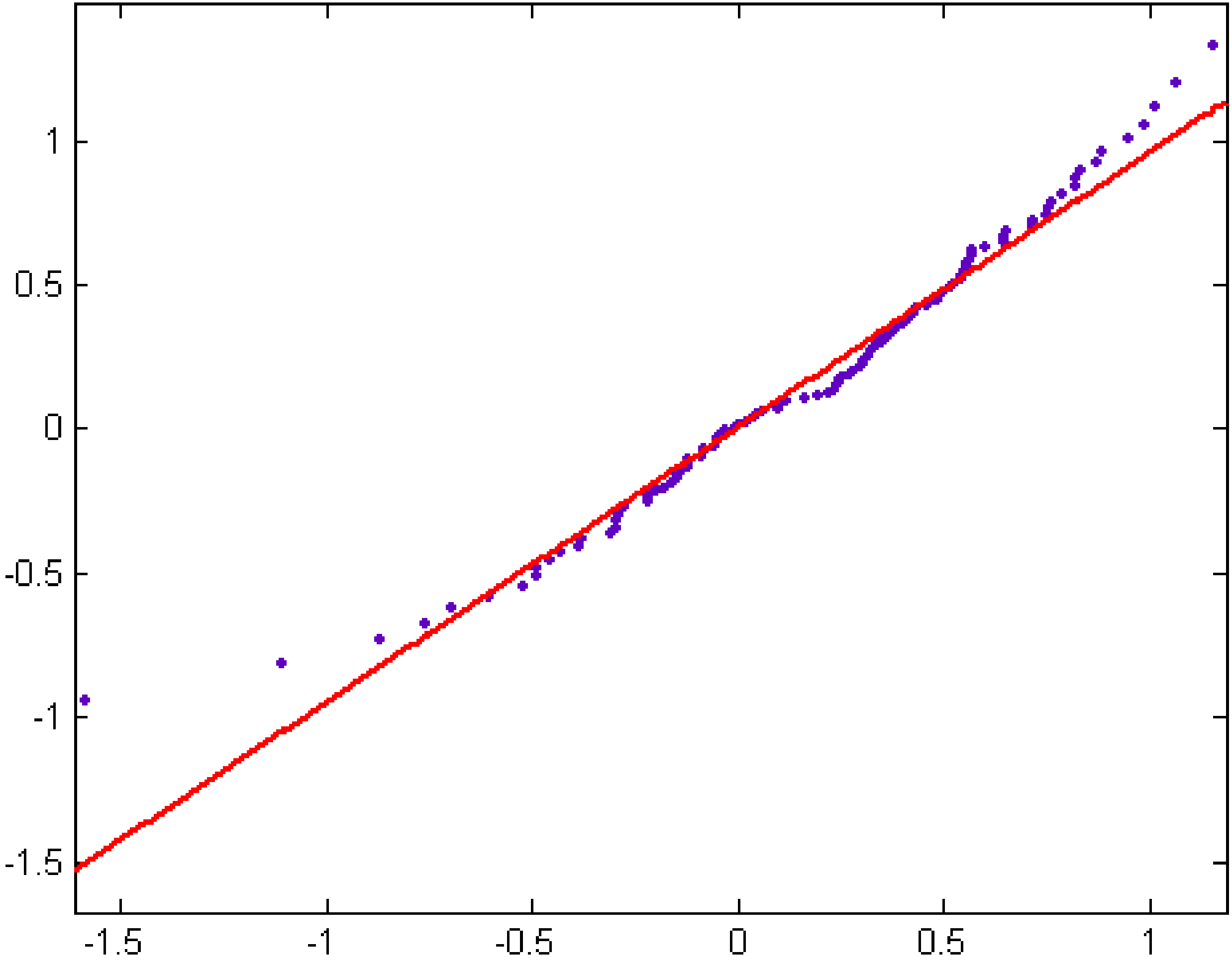


Figure 98 (n=p^13q, mean=0.1571, standard deviation=0.5139, size=200)
1.5
1
0.5
0
-0.5
-1
-1.5
-1.5
-1
-0.5
0
0.5
1

Figure 99 ($n=p^{14}q$, mean=0.1908, standard deviation=0.4797, size=111)



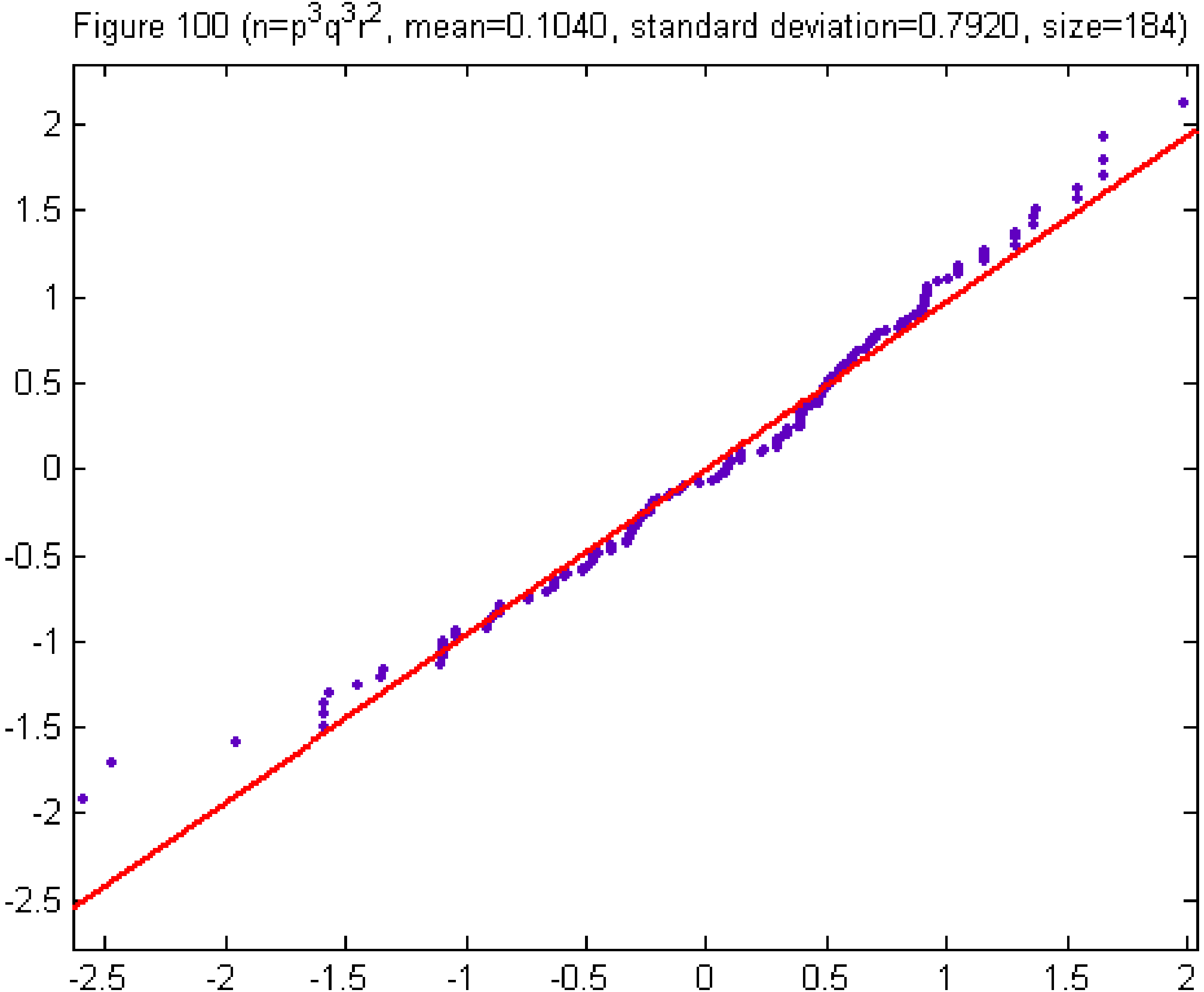
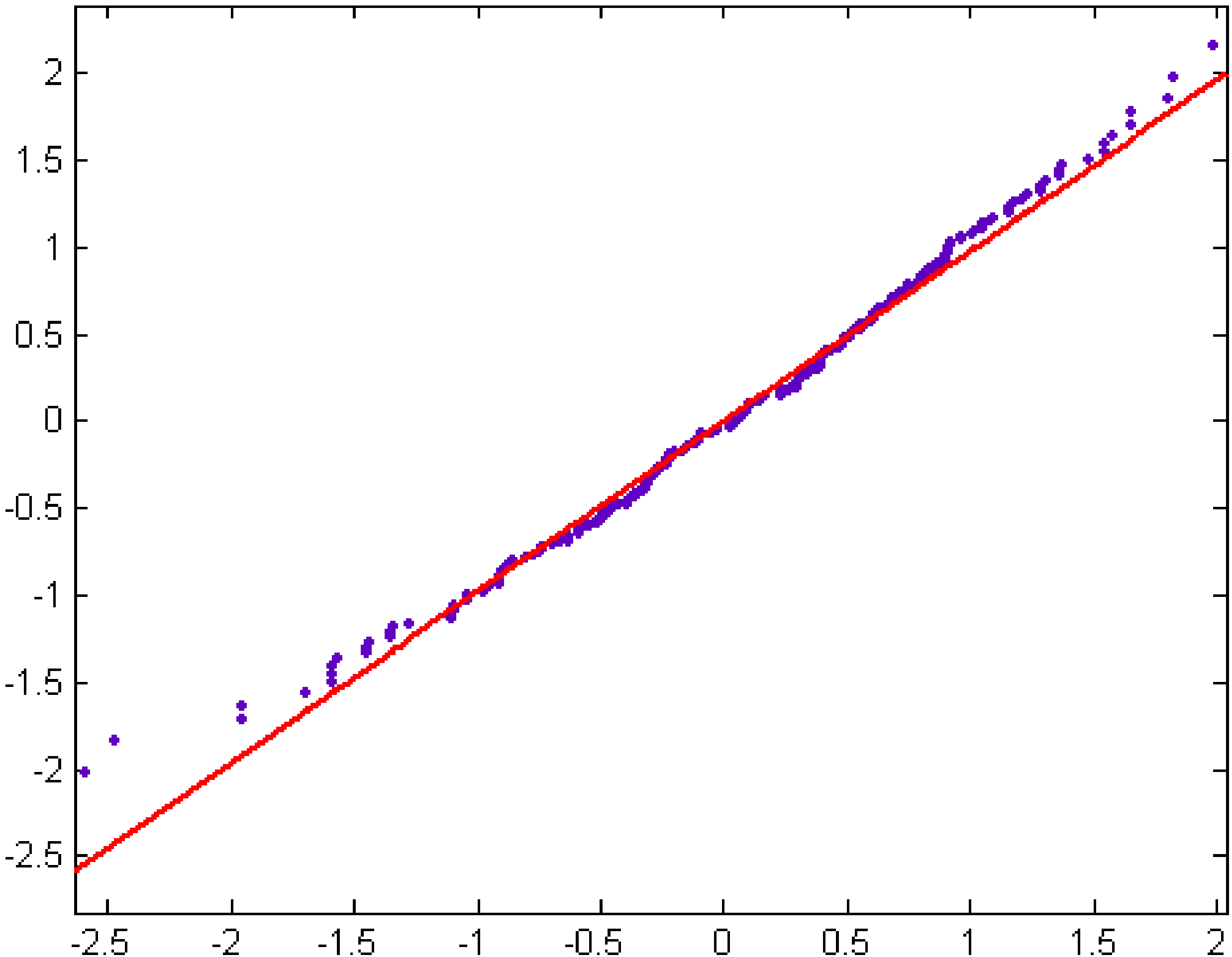


Figure 100 (n=p3q3r2, mean=0.1040, standard deviation=0.7920, size=184)
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
-2.5
-2.5
-2
-1.5
-1
-0.5
0
0.5
1
1.5
2

Figure 101 ($n=p^4q^2r^2$, mean=0.0674, standard deviation=0.7740, size=285)



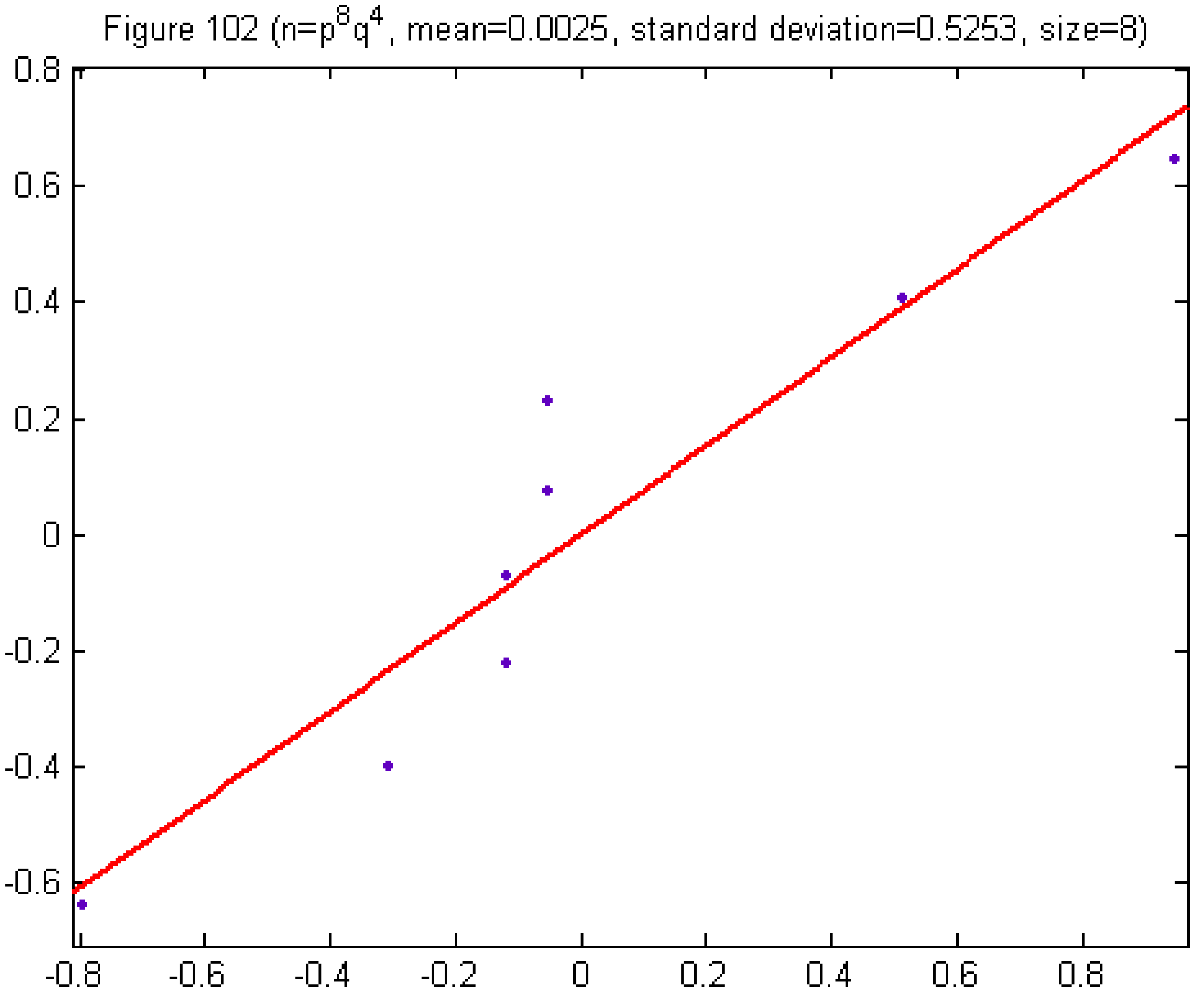


Figure 102 (n=p^8q^4, mean=0.0025, standard deviation=0.5253, size=8)
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
0.8

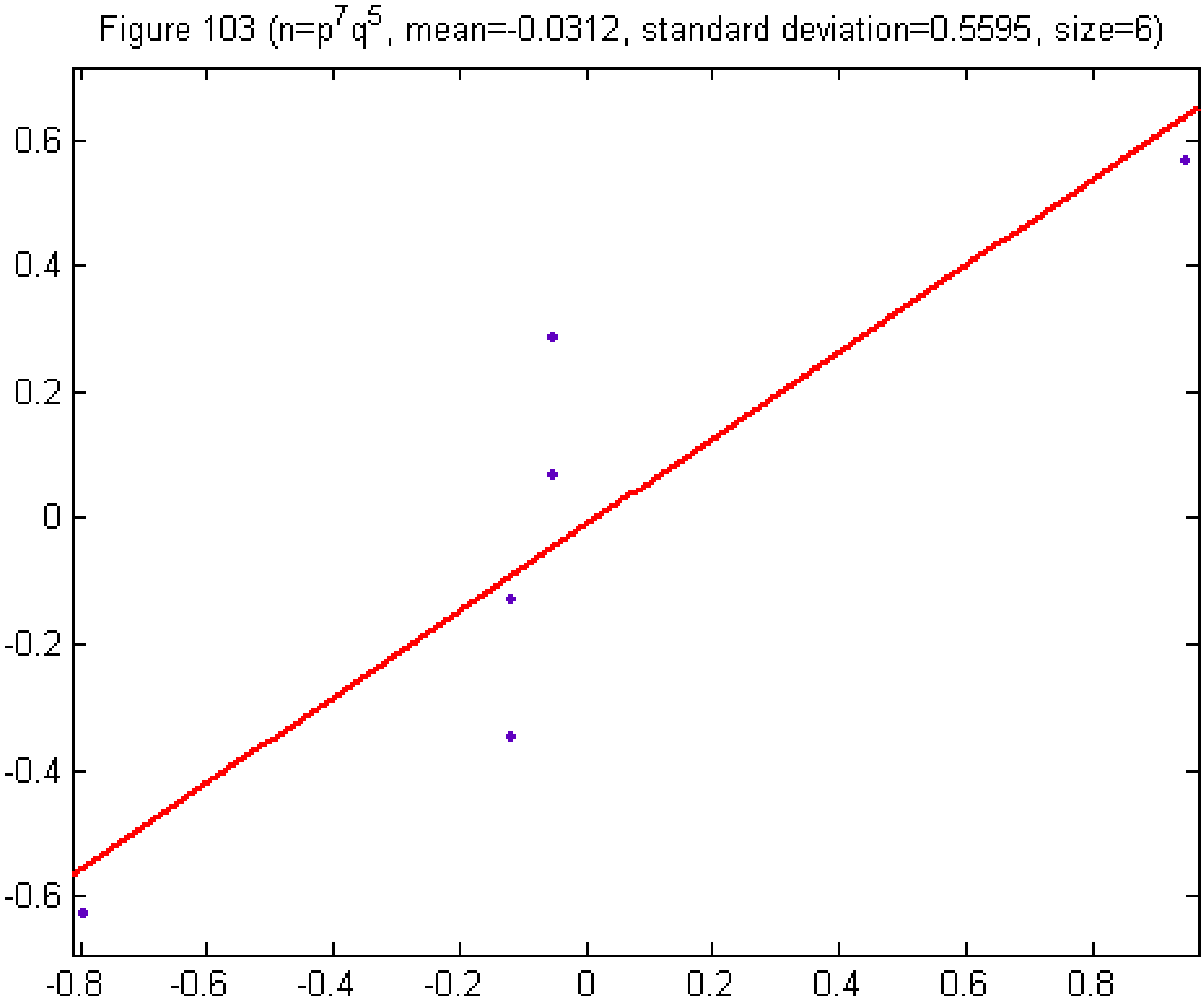
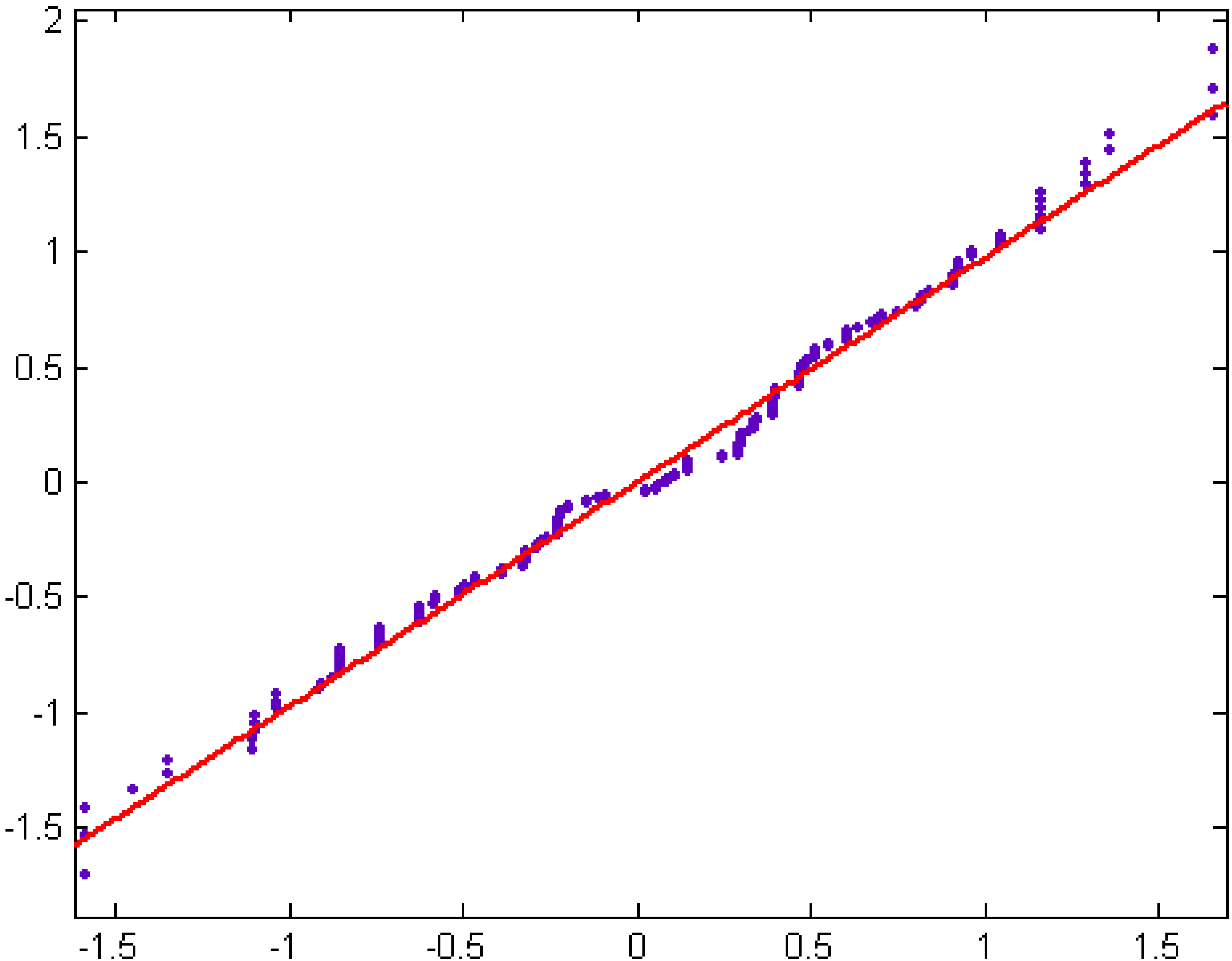


Figure 103 (n=p^7q^5, mean=-0.0312, standard deviation=0.5595, size=6)
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
0.8

Figure 104 ($n=p^4q^3r^2$, mean=0.0860, standard deviation=0.7003, size=188)



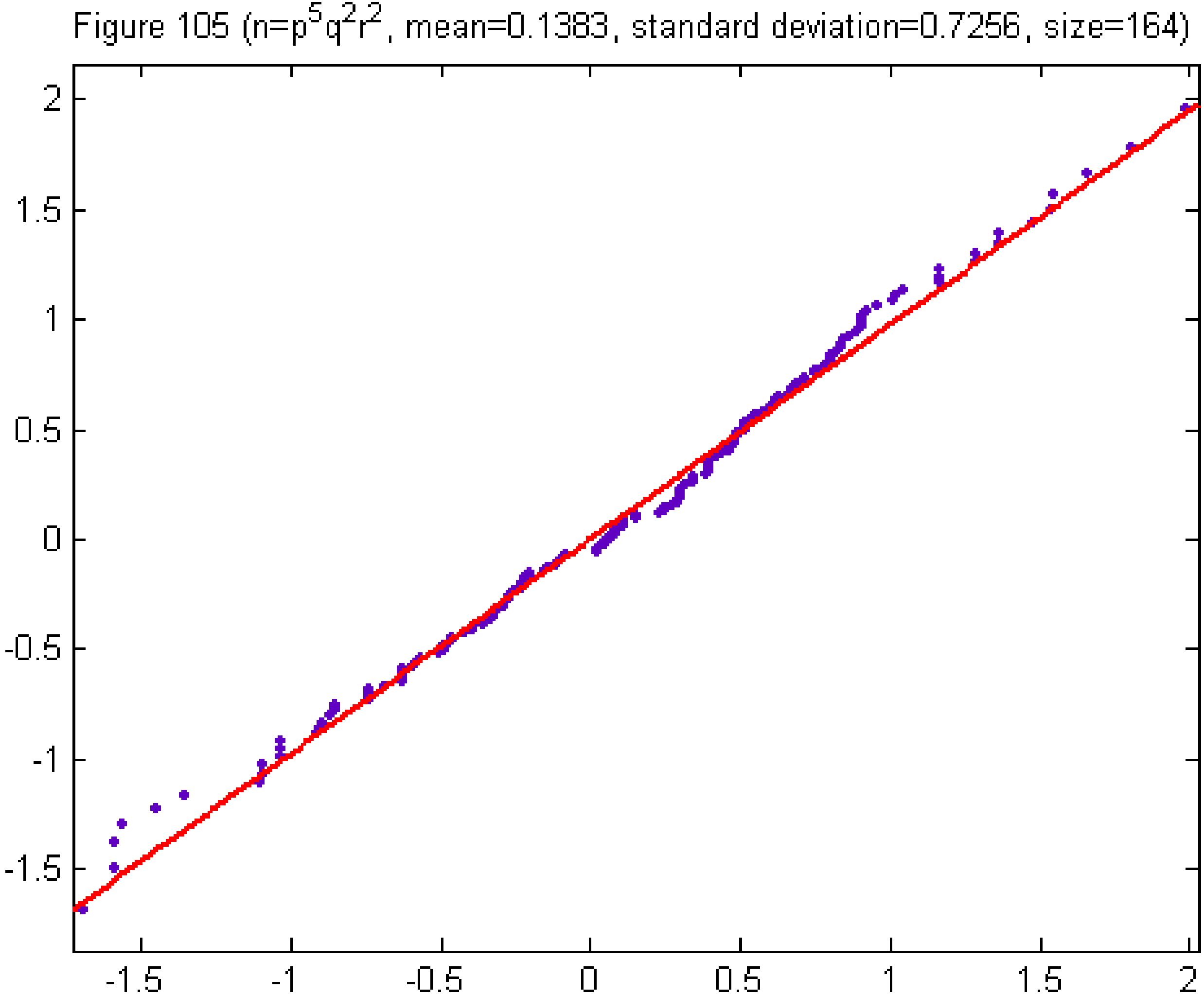
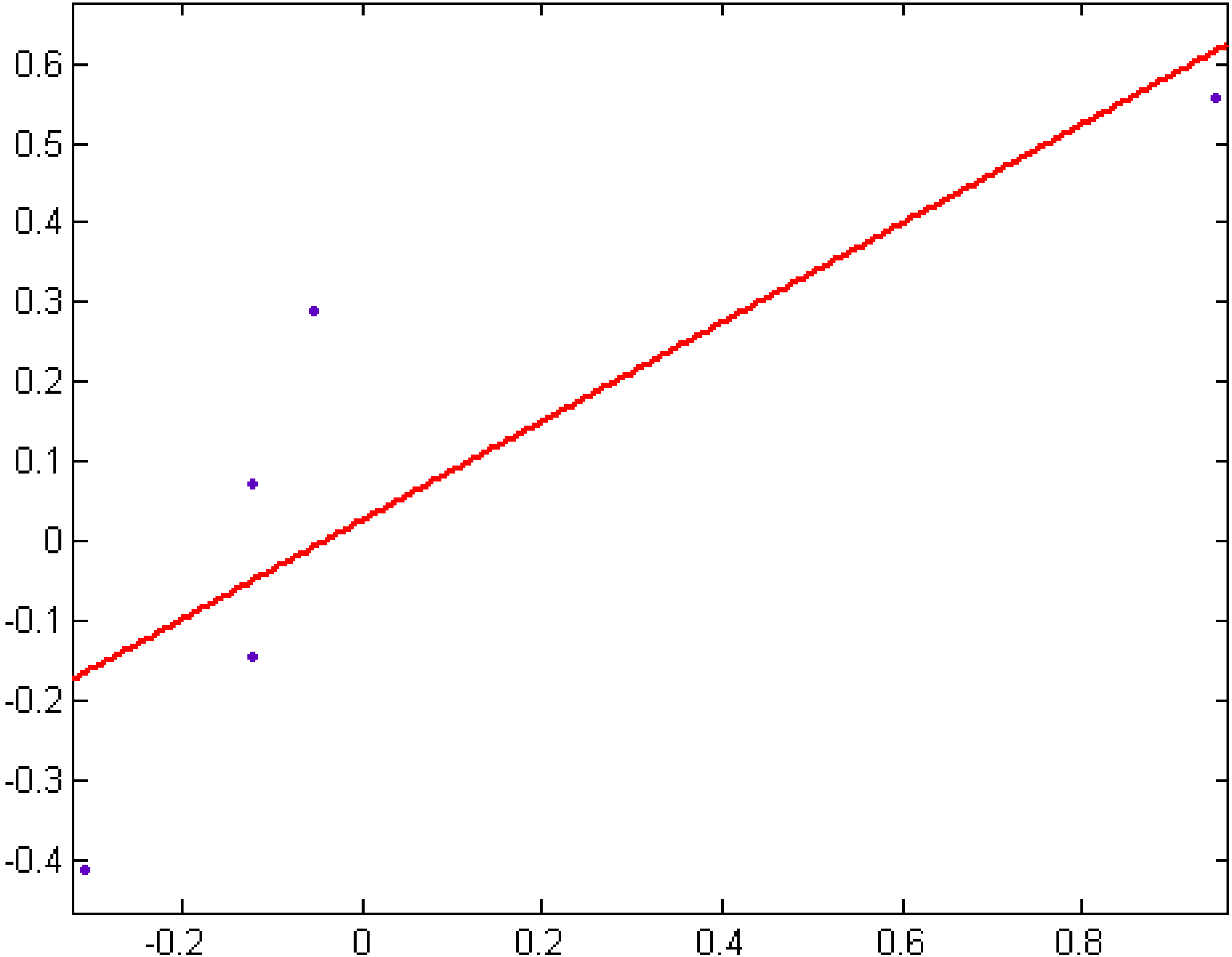


Figure 105 (n=p^5q^2r^2, mean=0.1383, standard deviation=0.7256, size=164)
2
1.5
1
0.5
0
-0.5
-1
-1.5
-1.5
-1
-0.5
0
0.5
1
1.5
2

Figure 106 ($n=p^9q^4$, mean=0.0704, standard deviation=0.5012, size=5)



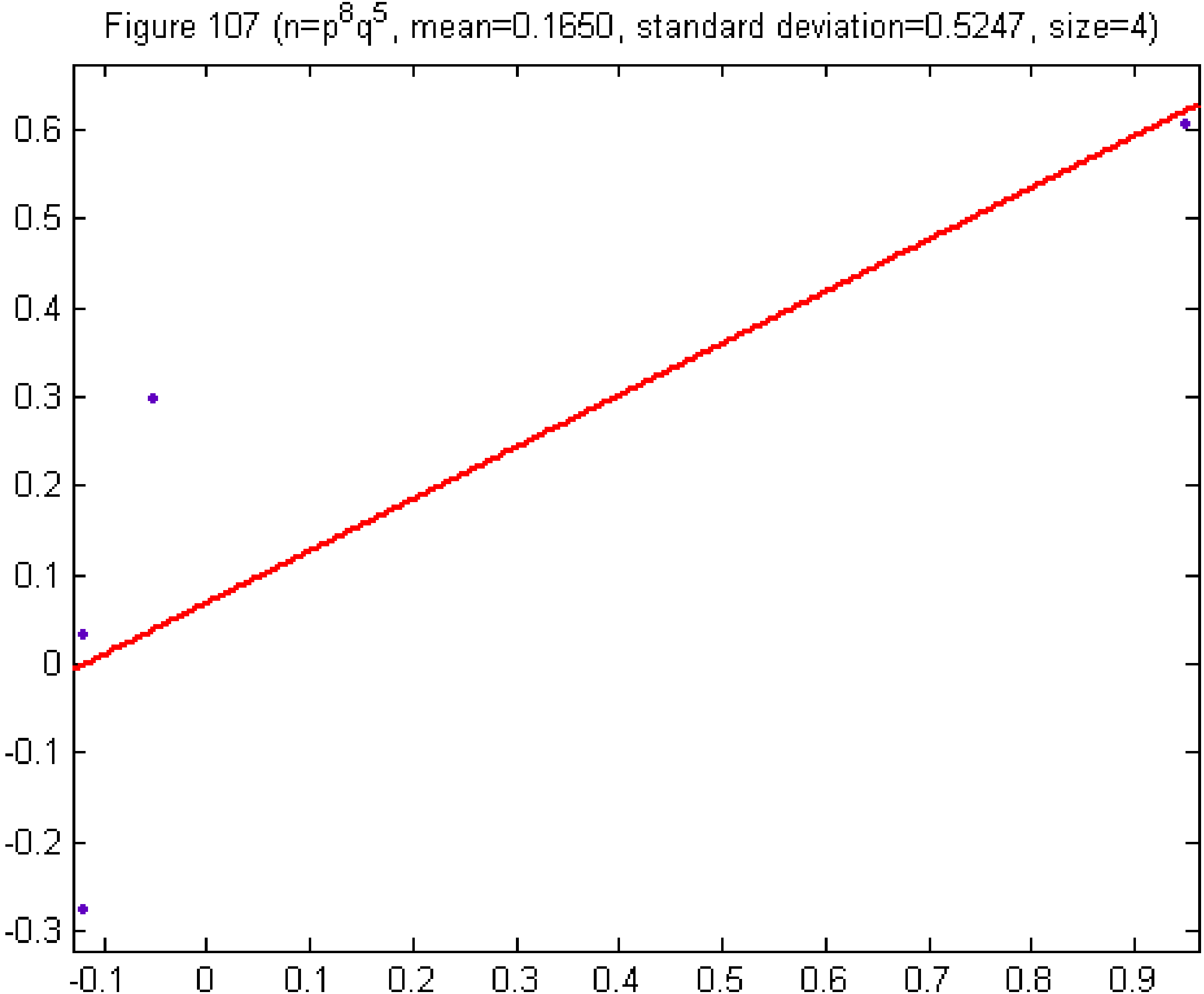


Figure 107 (n=p^8q^5, mean=0.1650, standard deviation=0.5247, size=4)
0.6
0.5
0.4
0.3
0.2
0.1
0
-0.1
-0.2
-0.3
-0.1
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9

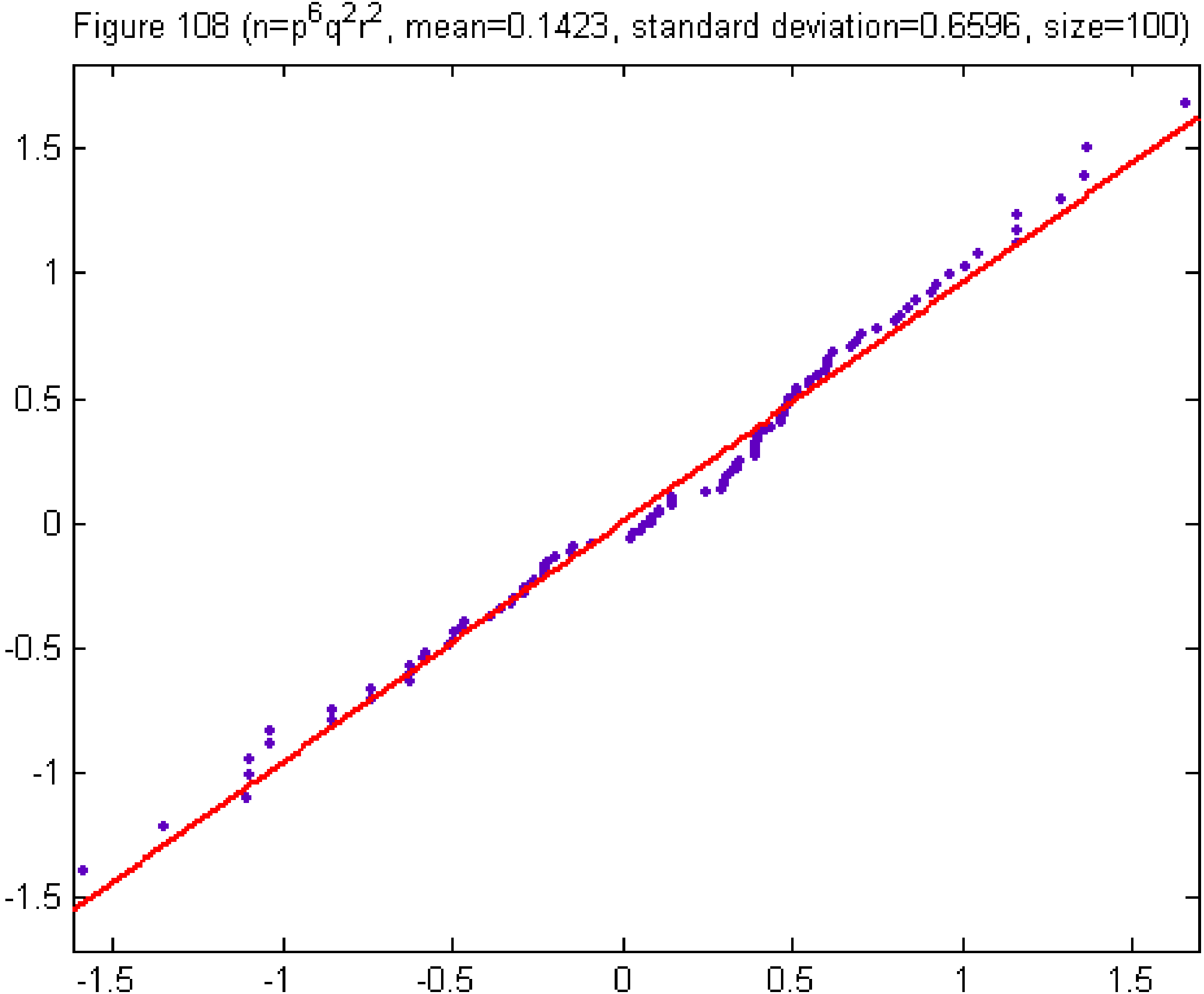


Figure 108 ($n=p^6q^2r^2$, mean=0.1423, standard deviation=0.6596, size=100)
1.5
1
0.5
0
-0.5
-1
-1.5
-1.5
-1
-0.5
0
0.5
1
1.5

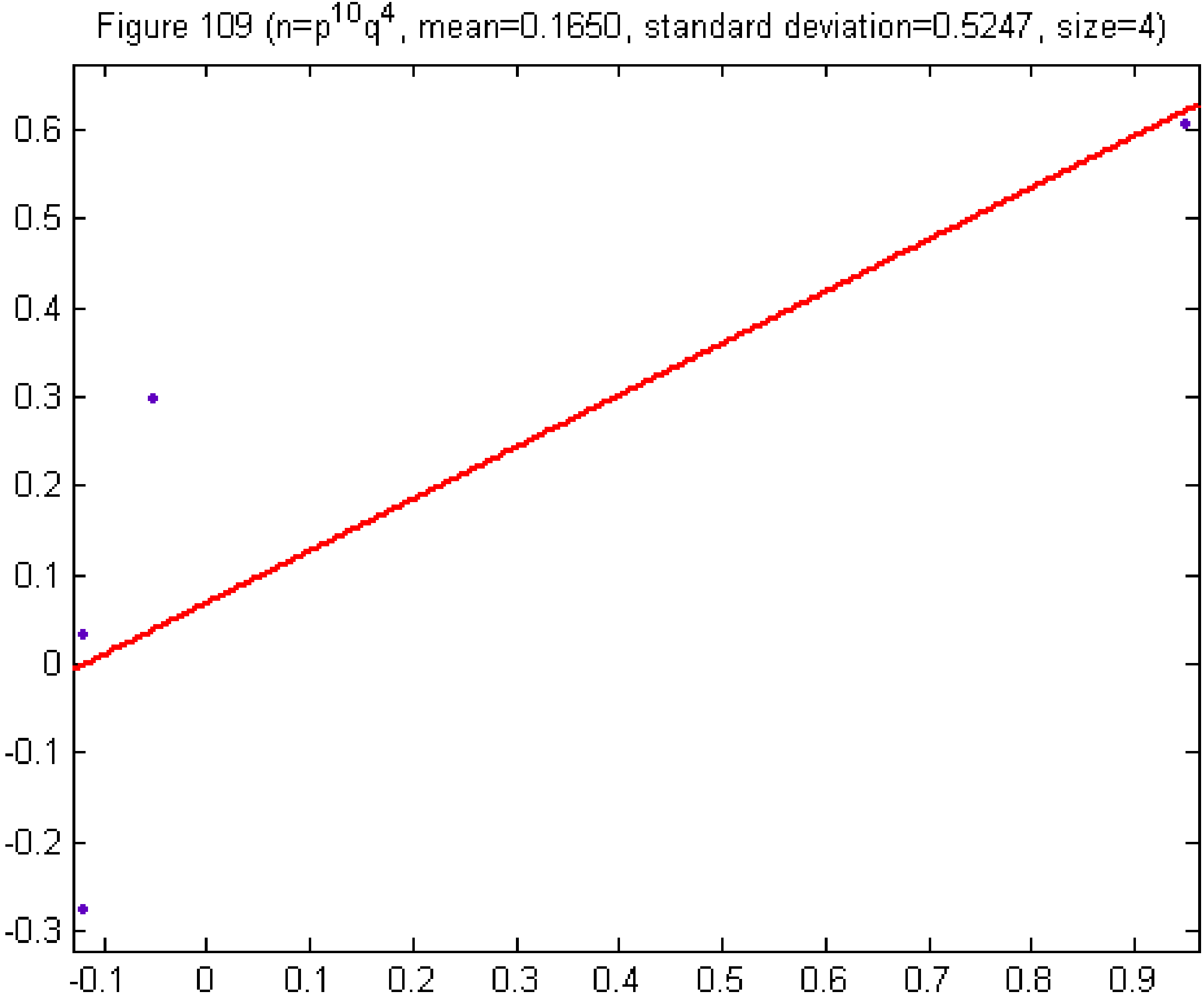


Figure 109 (n=p^10q^4, mean=0.1650, standard deviation=0.5247, size=4)
0.6
0.5
0.4
0.3
0.2
0.1
0
-0.1
-0.2
-0.3
-0.1
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9

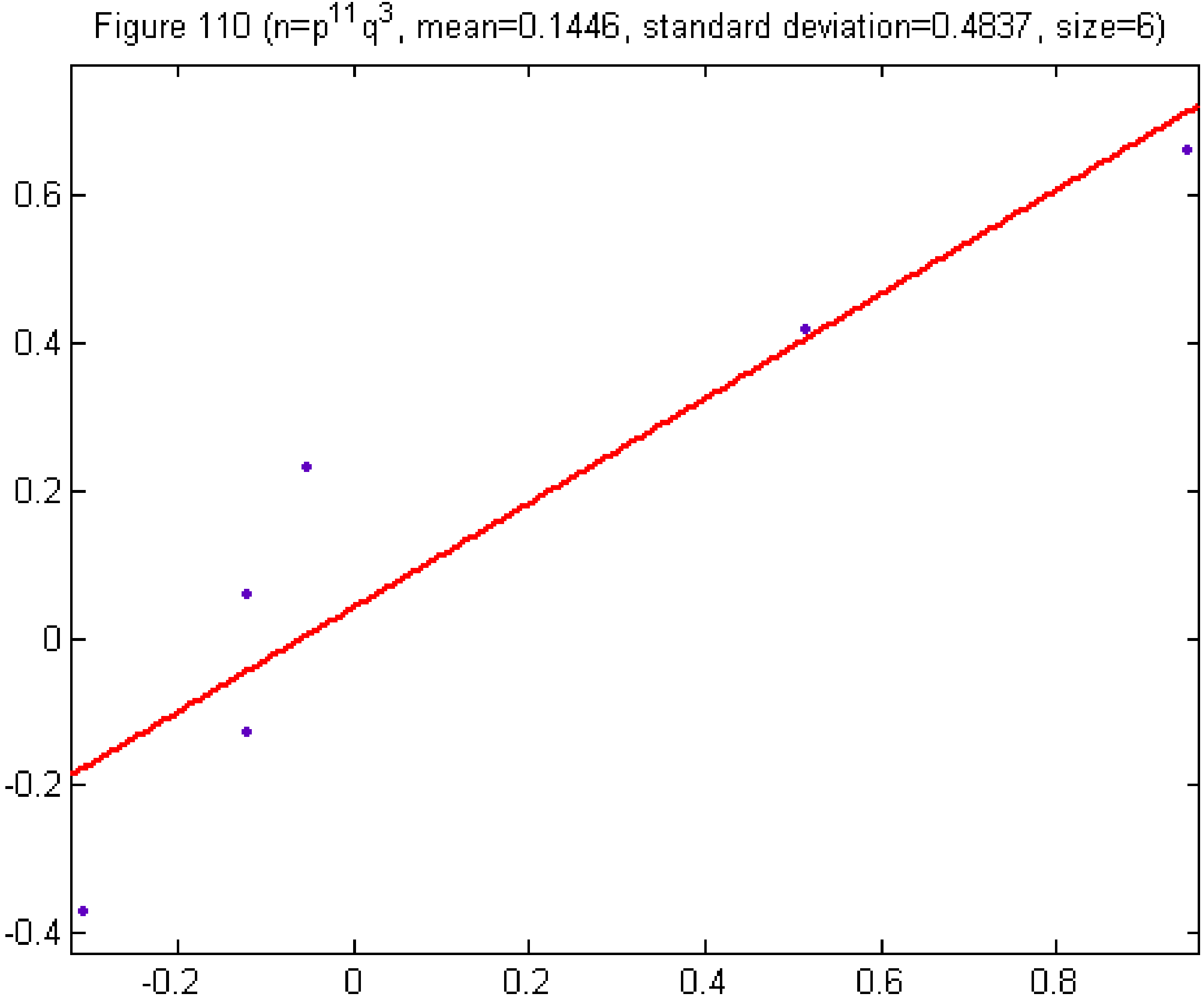


Figure 110 (n=p11q3, mean=0.1446, standard deviation=0.4837, size=6)
0.6
0.4
0.2
0
-0.2
-0.4
-0.2
0
0.2
0.4
0.6
0.8

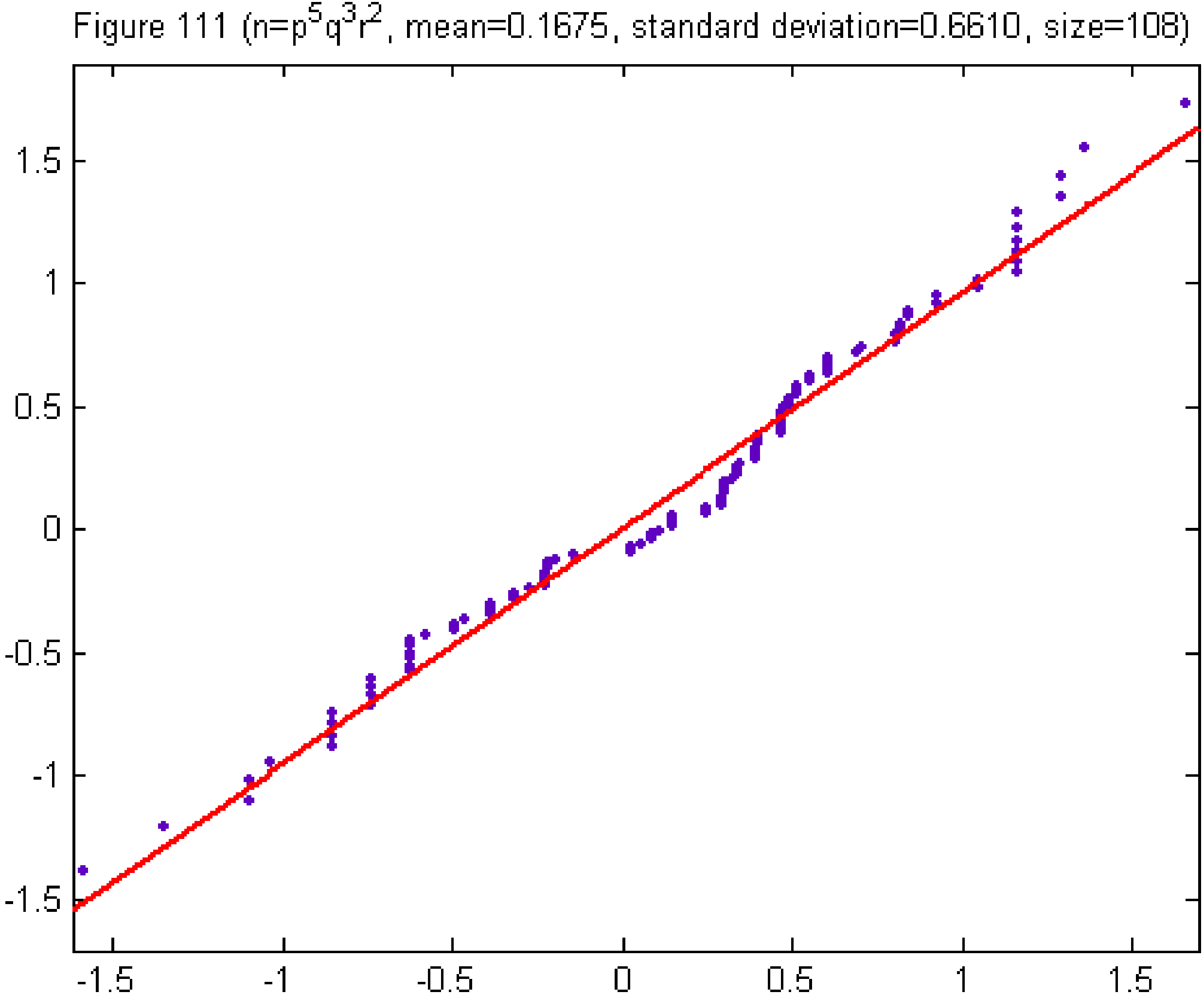


Figure 111 (n=p^5q^3r^2, mean=0.1675, standard deviation=0.6610, size=108)
1.5
1
0.5
0
-0.5
-1
-1.5
-1.5
-1
-0.5
0
0.5
1
1.5

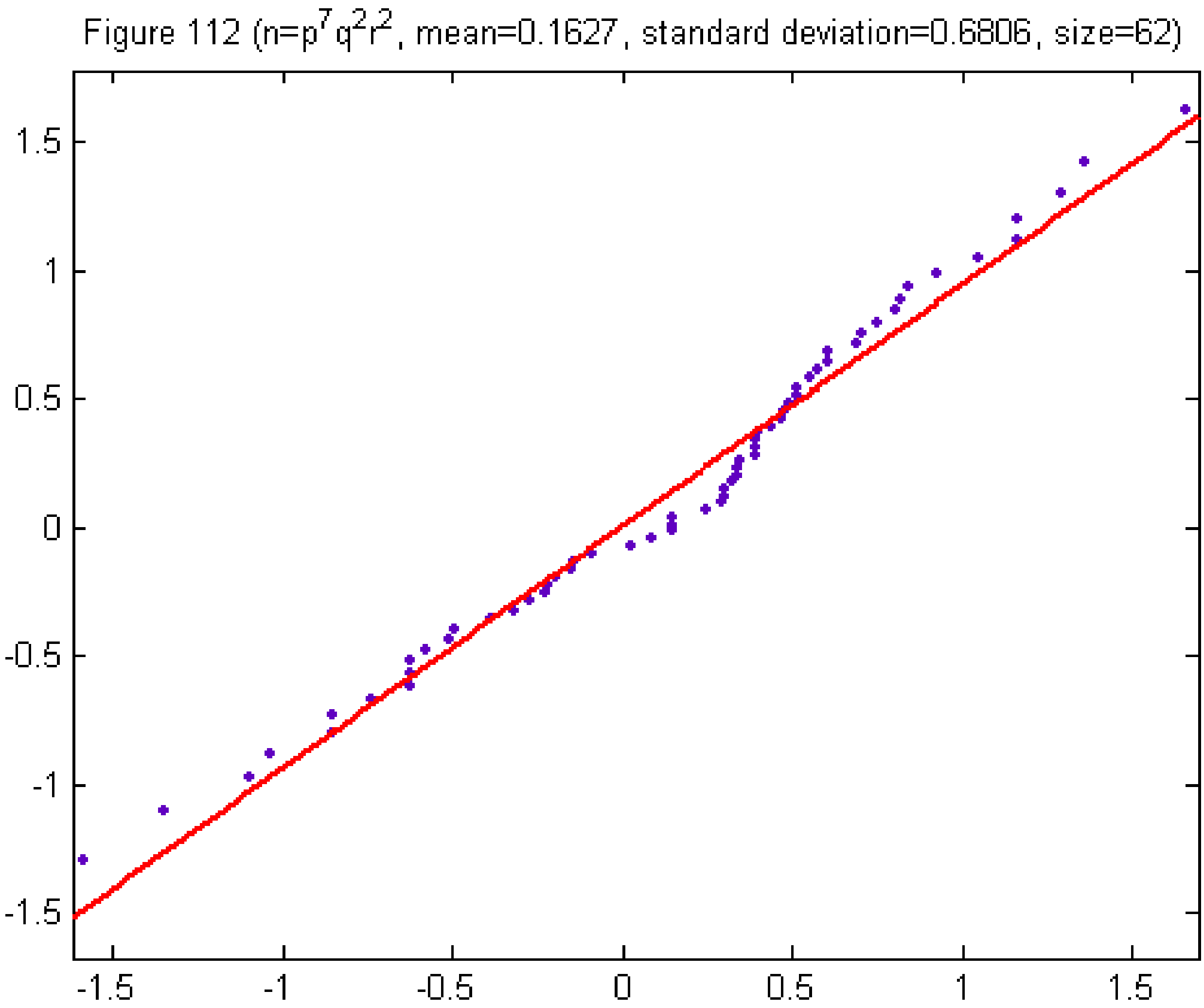
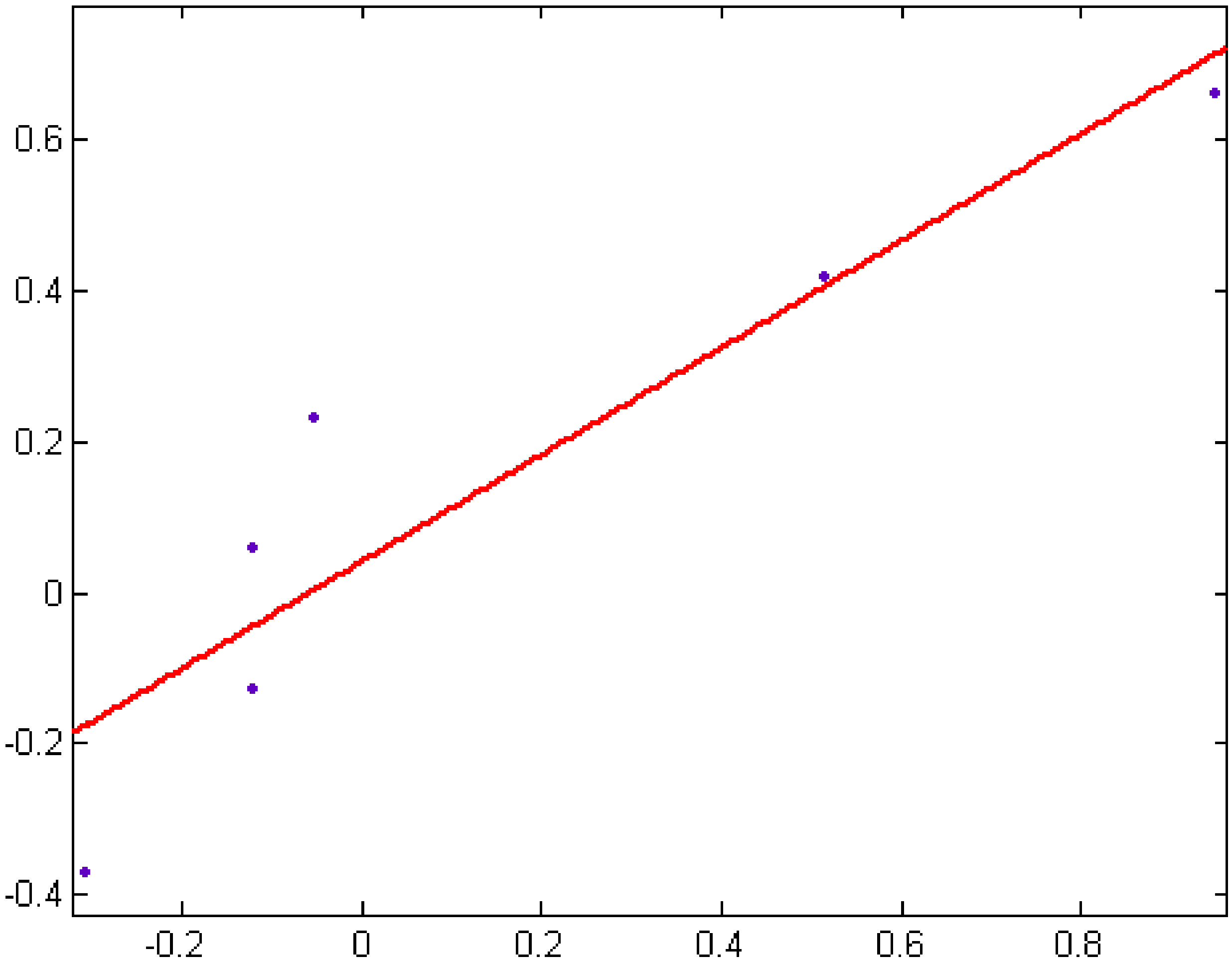


Figure 112 (n=p^7q^2r^2, mean=0.1627, standard deviation=0.6806, size=62)
1.5
1
0.5
0
-0.5
-1
-1.5
-1.5
-1
-0.5
0
0.5
1
1.5

Figure 113 ($n=p^{12}q^3$, mean=0.1446, standard deviation=0.4837, size=6)



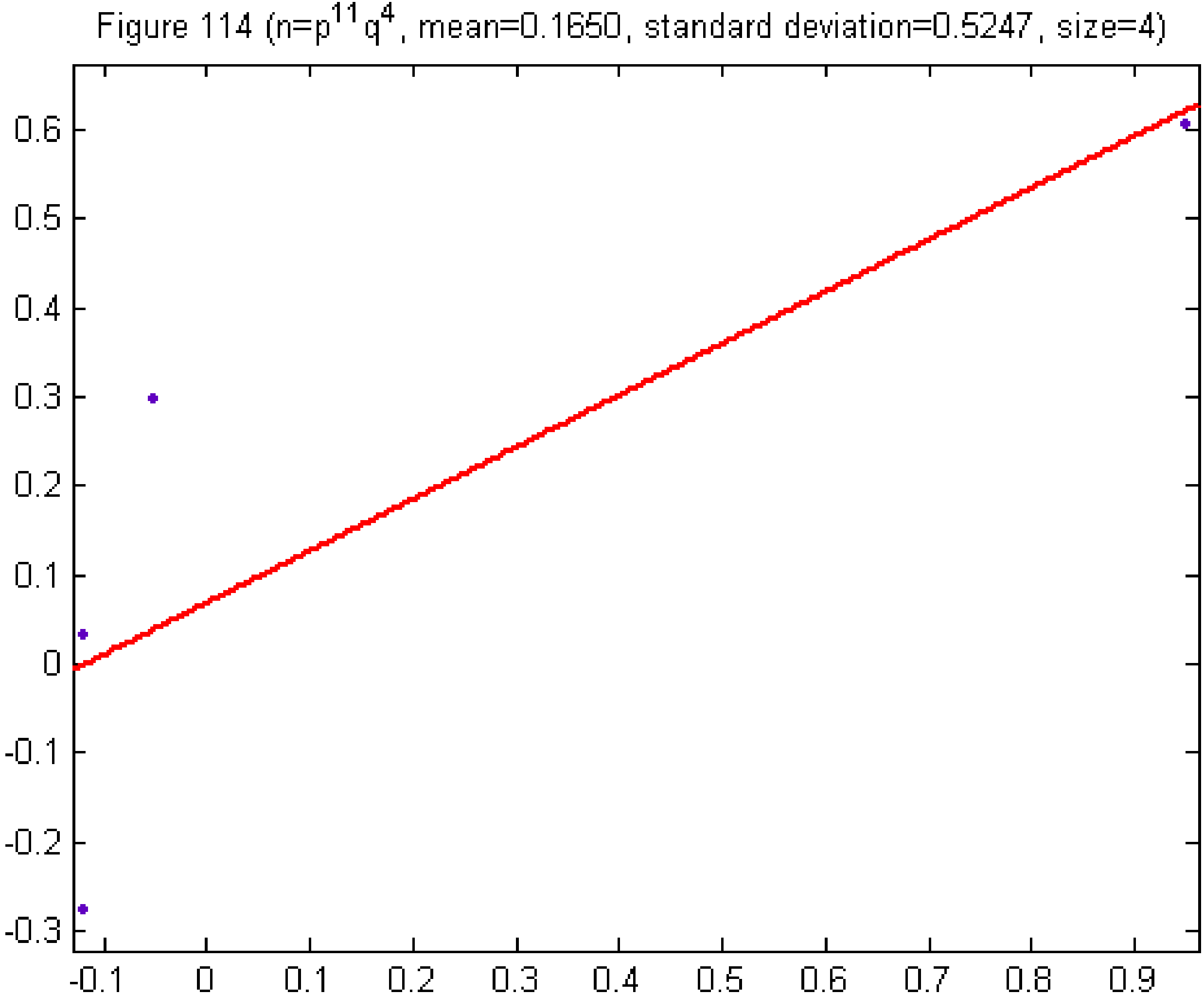


Figure 114 (n=p^11 q^4, mean=0.1650, standard deviation=0.5247, size=4)
0.6
0.5
0.4
0.3
0.2
0.1
0
-0.1
-0.2
-0.3
-0.1
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9

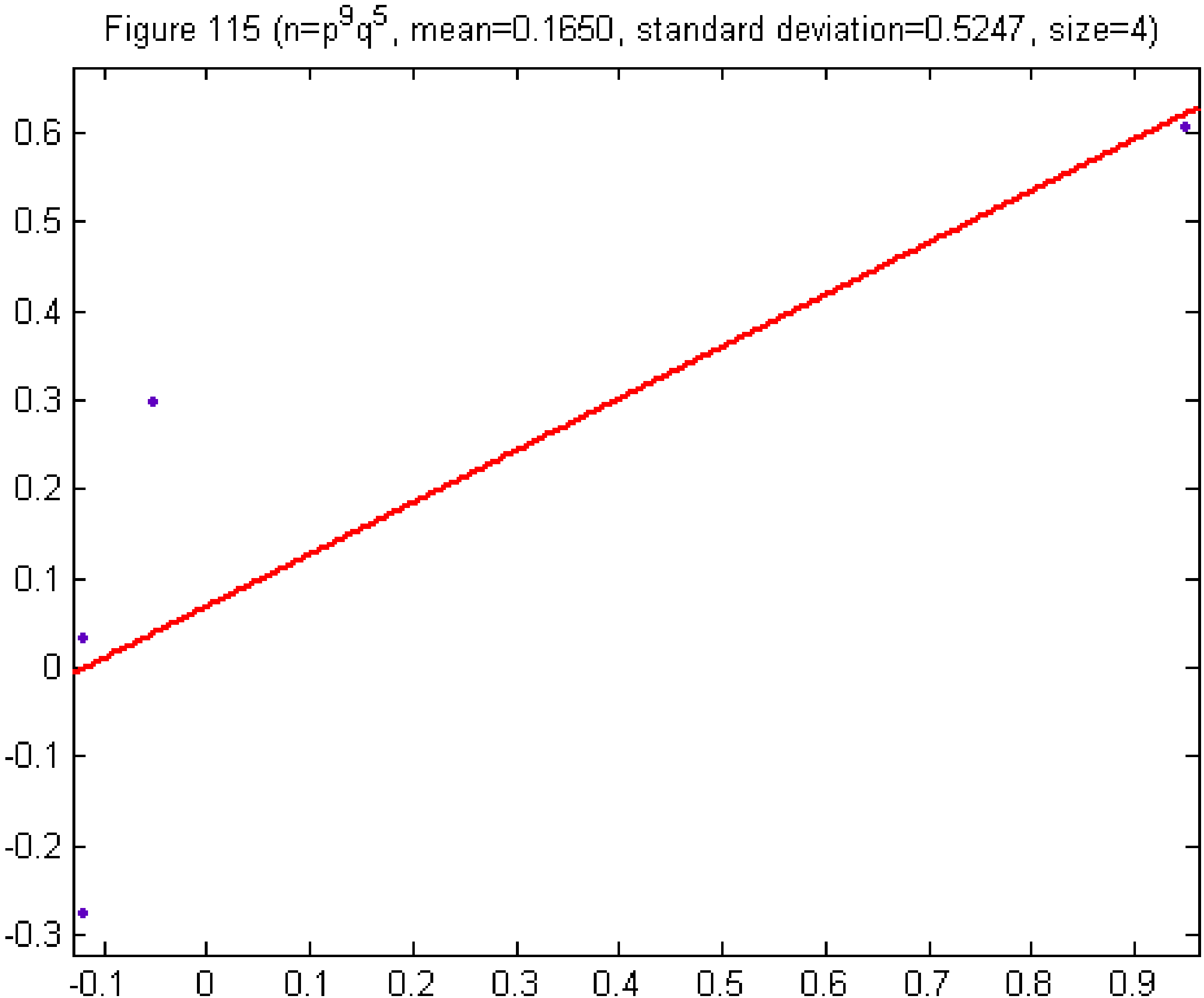
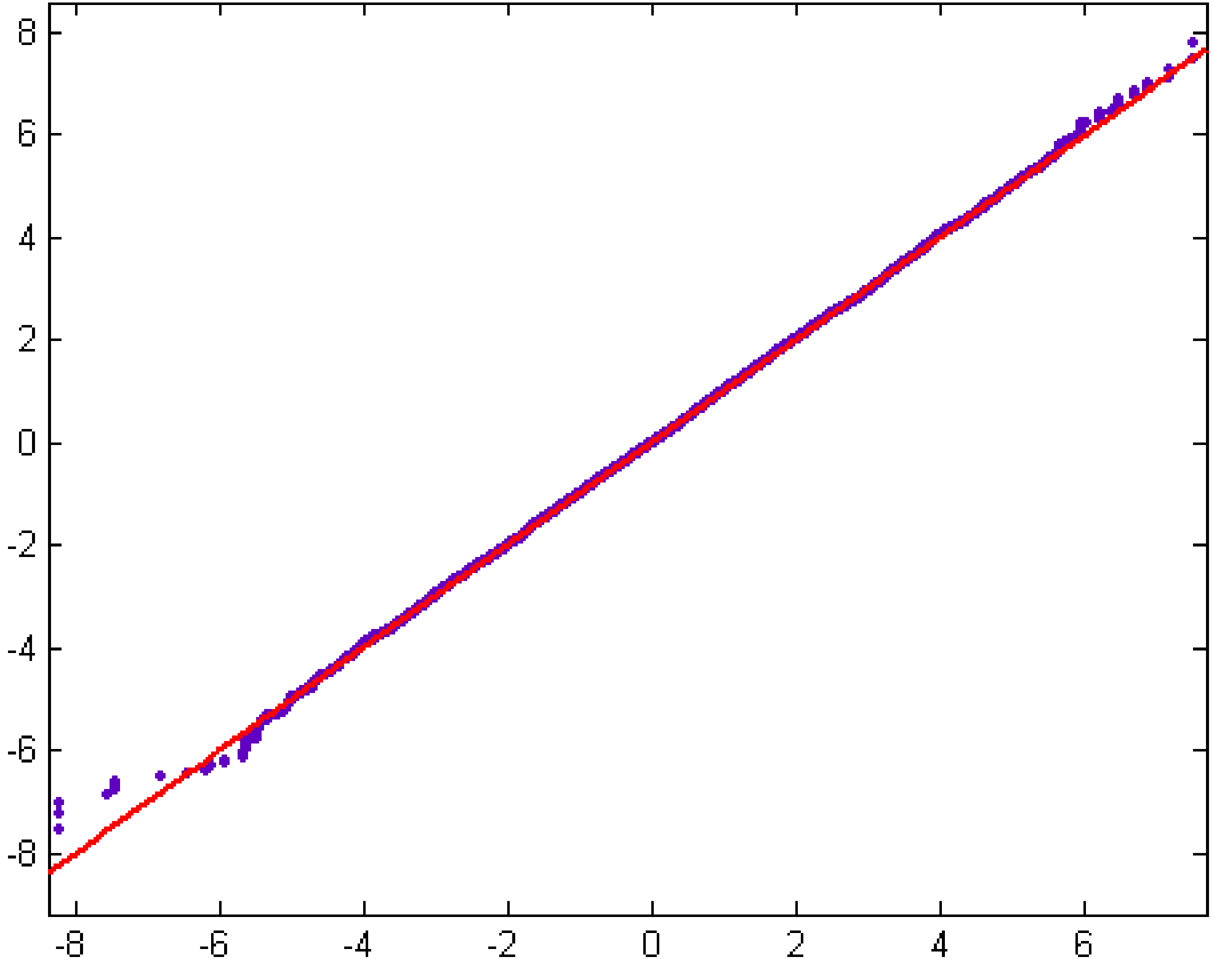


Figure 115 (n=p9q5, mean=0.1650, standard deviation=0.5247, size=4)
0.6
0.5
0.4
0.3
0.2
0.1
0
-0.1
-0.2
-0.3
-0.1
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9

Figure 116 ($n=p^3q^2rst$, mean=0.1078, standard deviation=1.9380, size=25830)



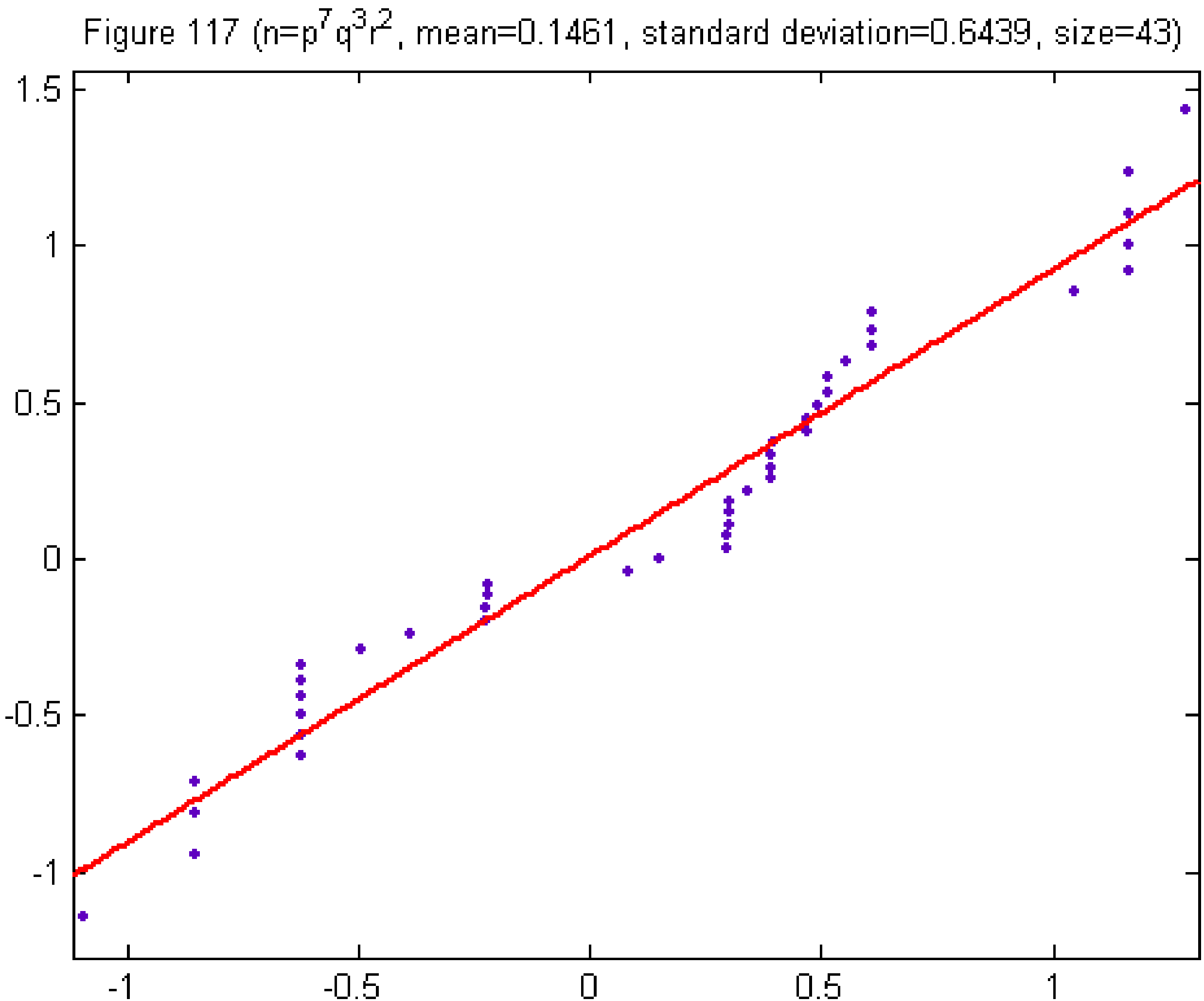


Figure 117 (n=p^7q^3r^2, mean=0.1461, standard deviation=0.6439, size=43)
1.5
1
0.5
0
-0.5
-1
-1
-0.5
0
0.5
1

Figure 118 ($n=p^4qrst$, mean=0.2232, standard deviation=1.9457, size=24192)

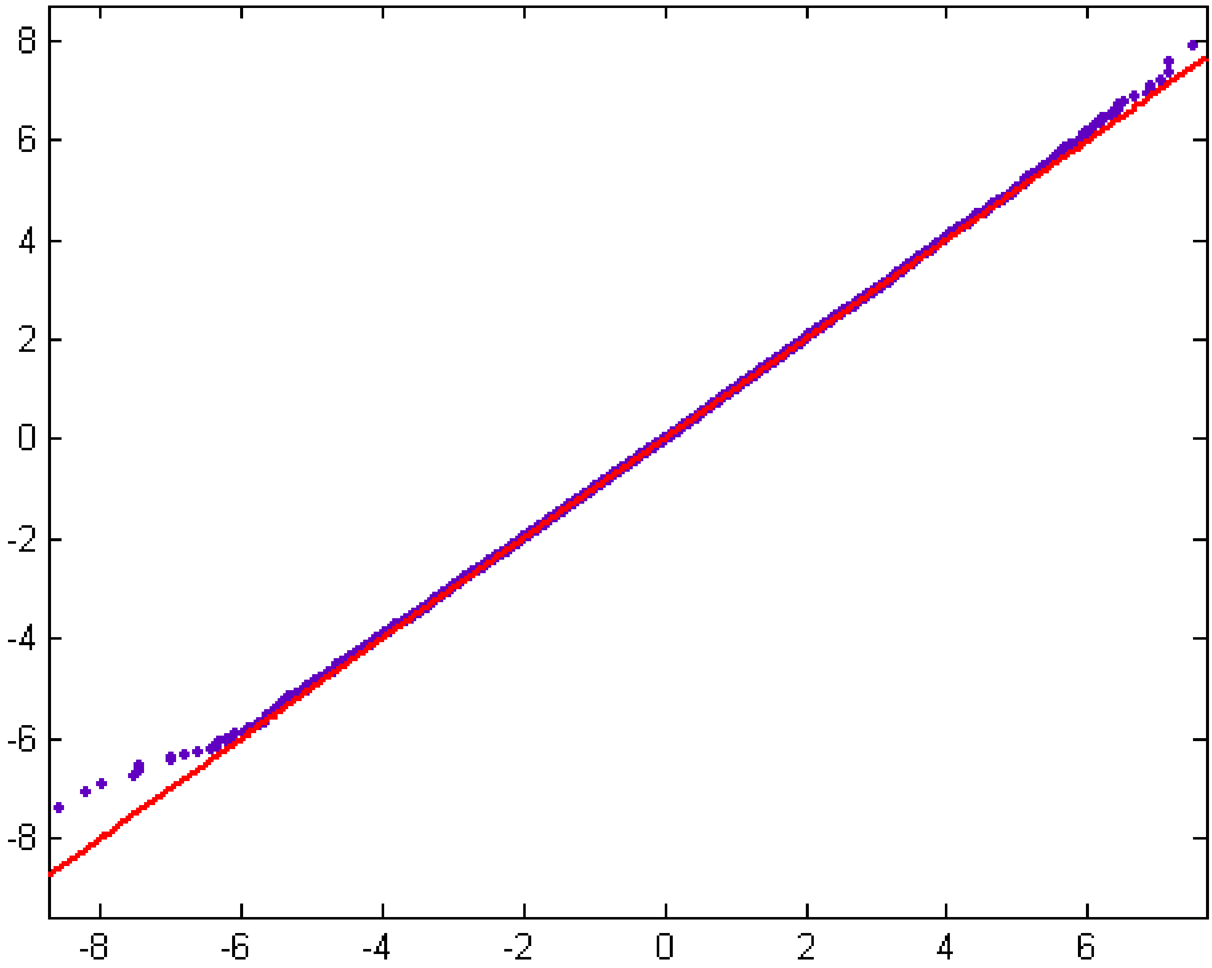


Figure 119 (n=p^2q^2r^2st, mean=-0.0402, standard deviation=1.8958, size=4052)

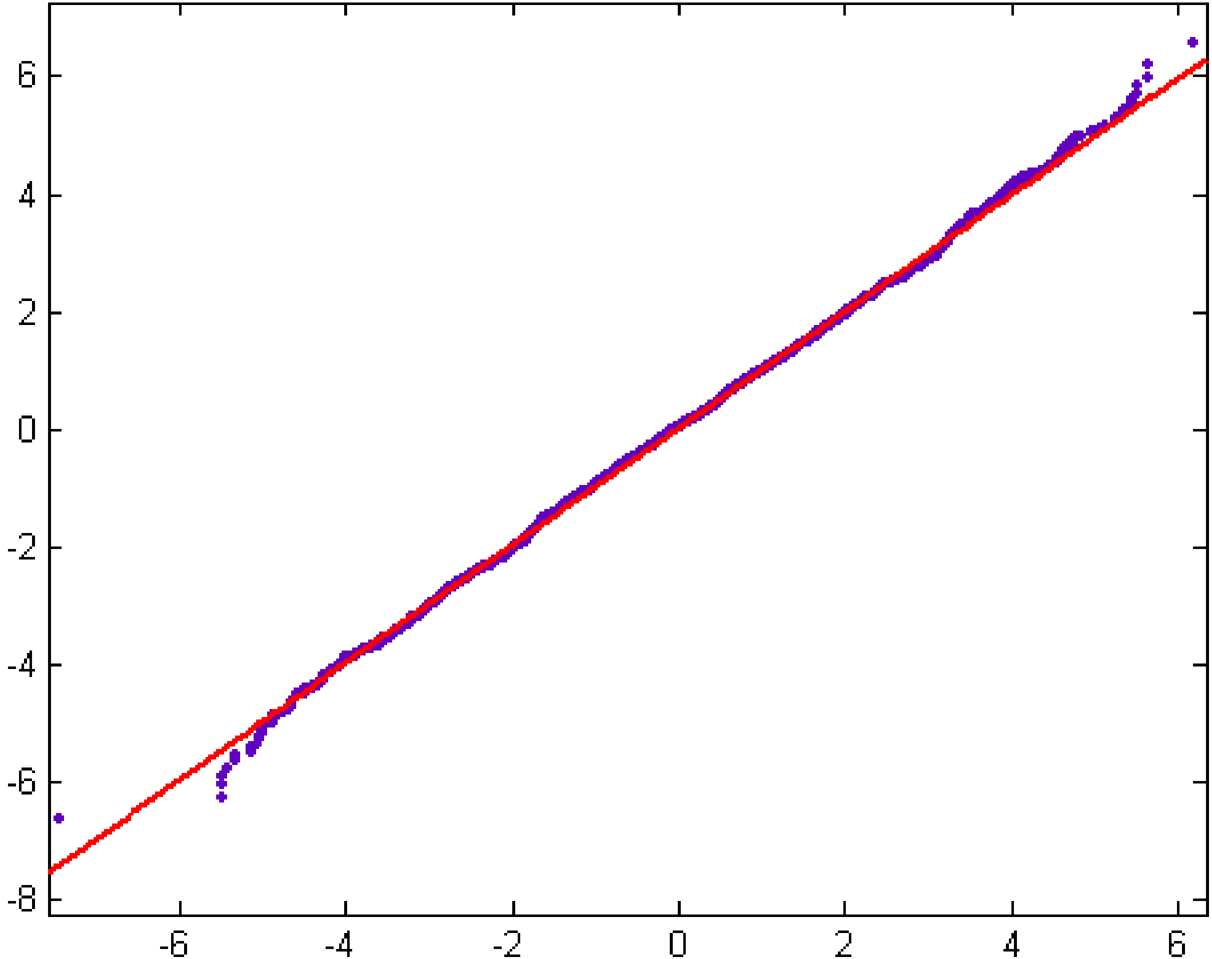
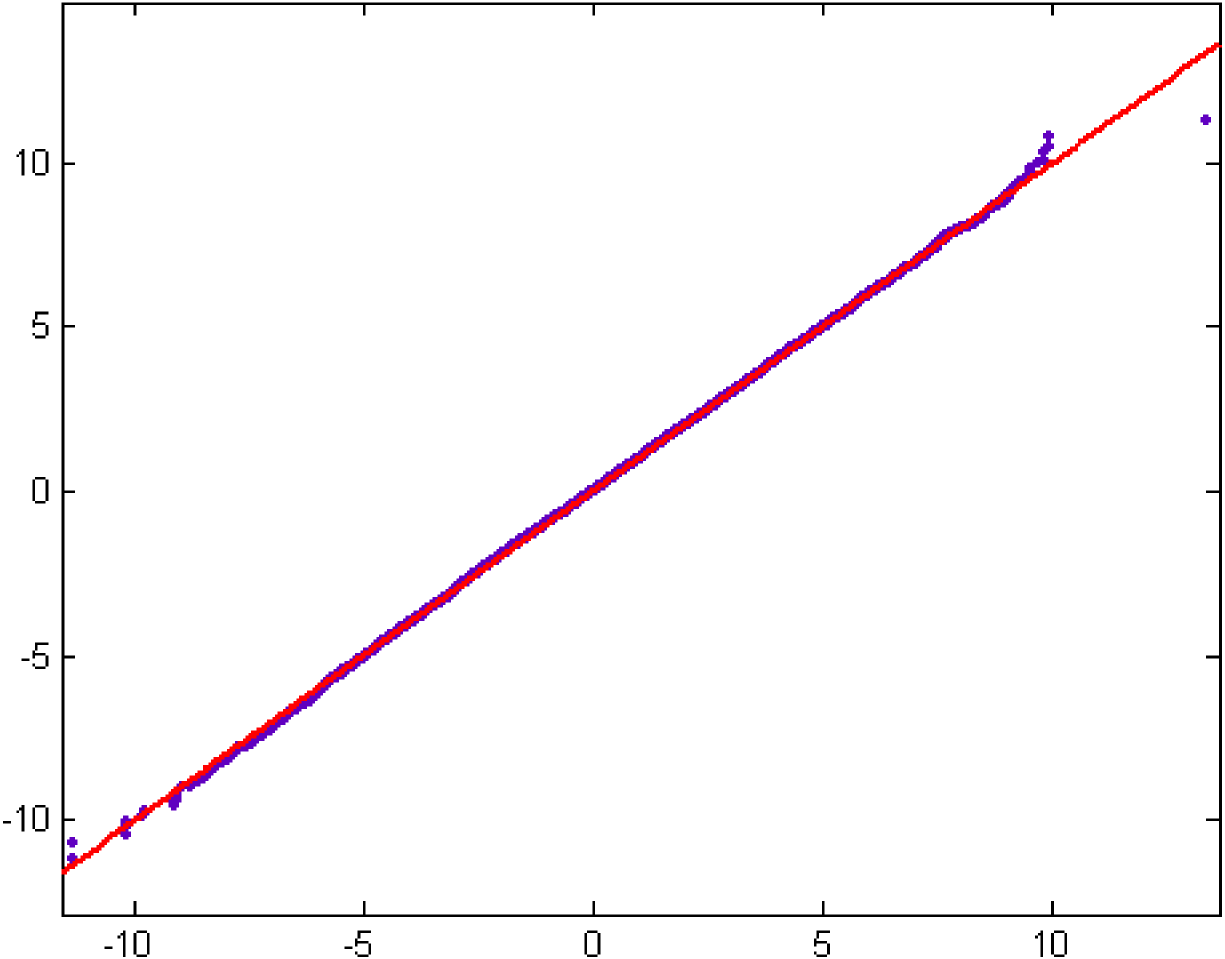


Figure 120 ($n=p^2qrstu$, mean=0.0196, standard deviation=2.8629, size=22852)



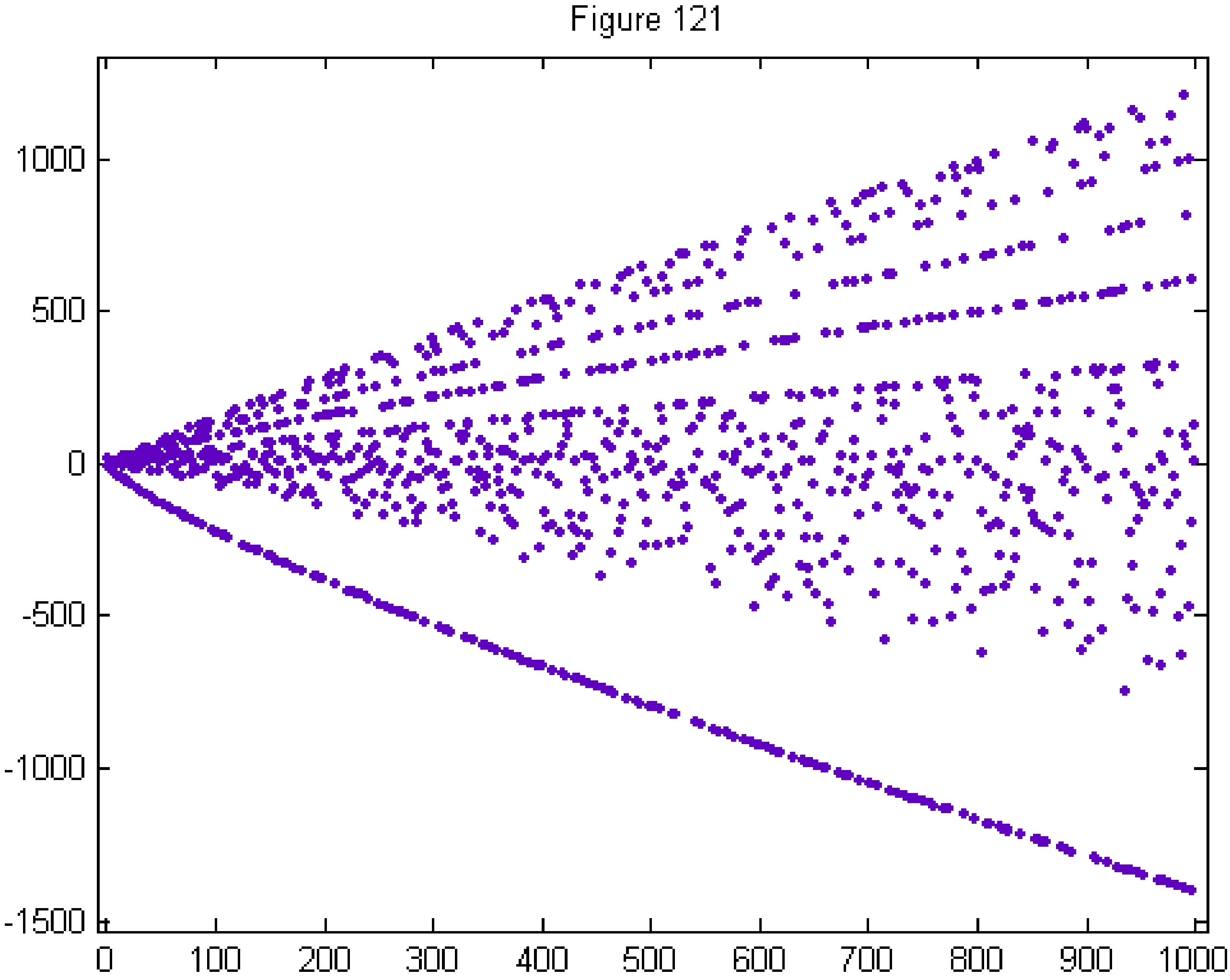
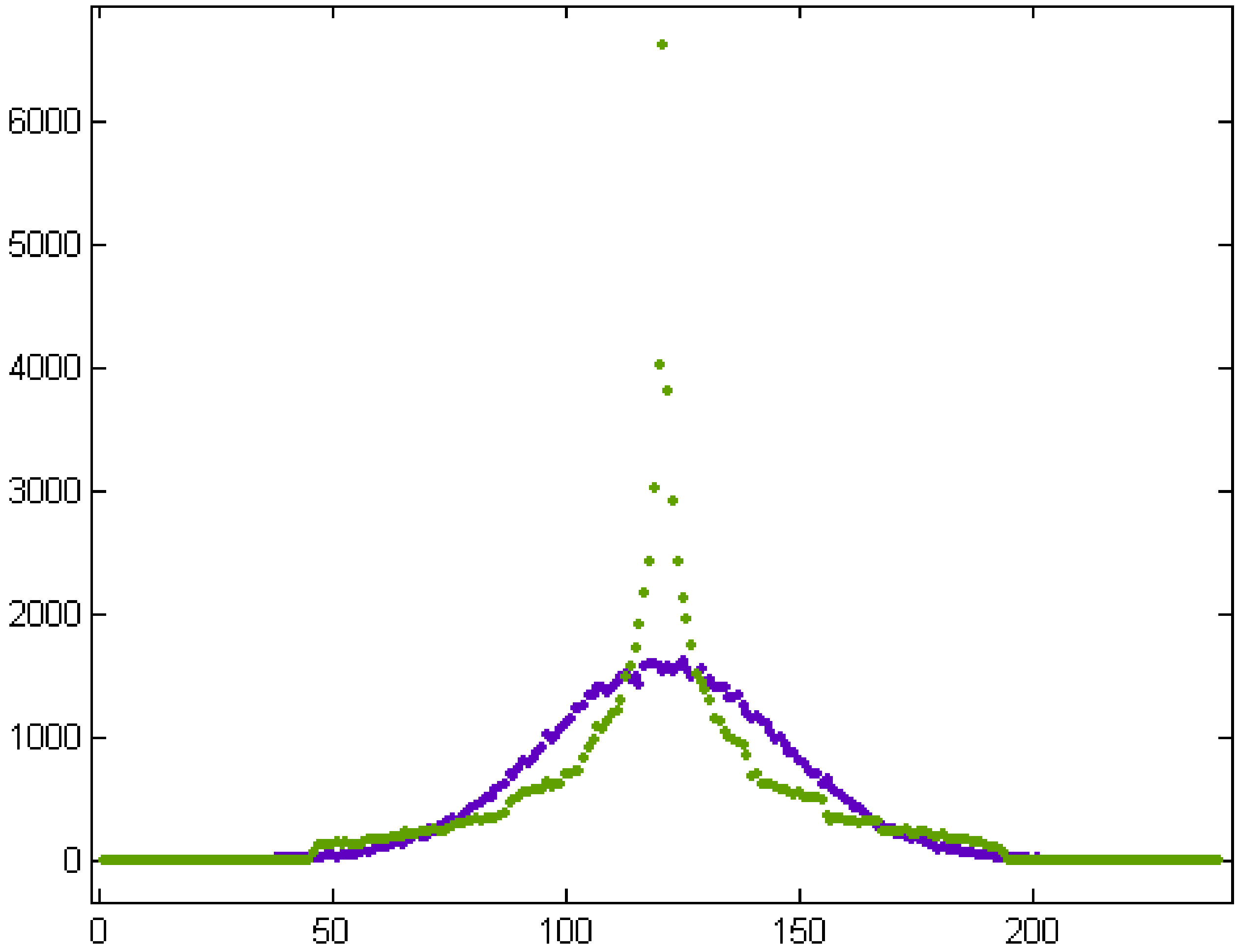


Figure 121
1000
500
0
-500
-1000
-1500
0
100
200
300
400
500
600
700
800
900
1000

Figure 122



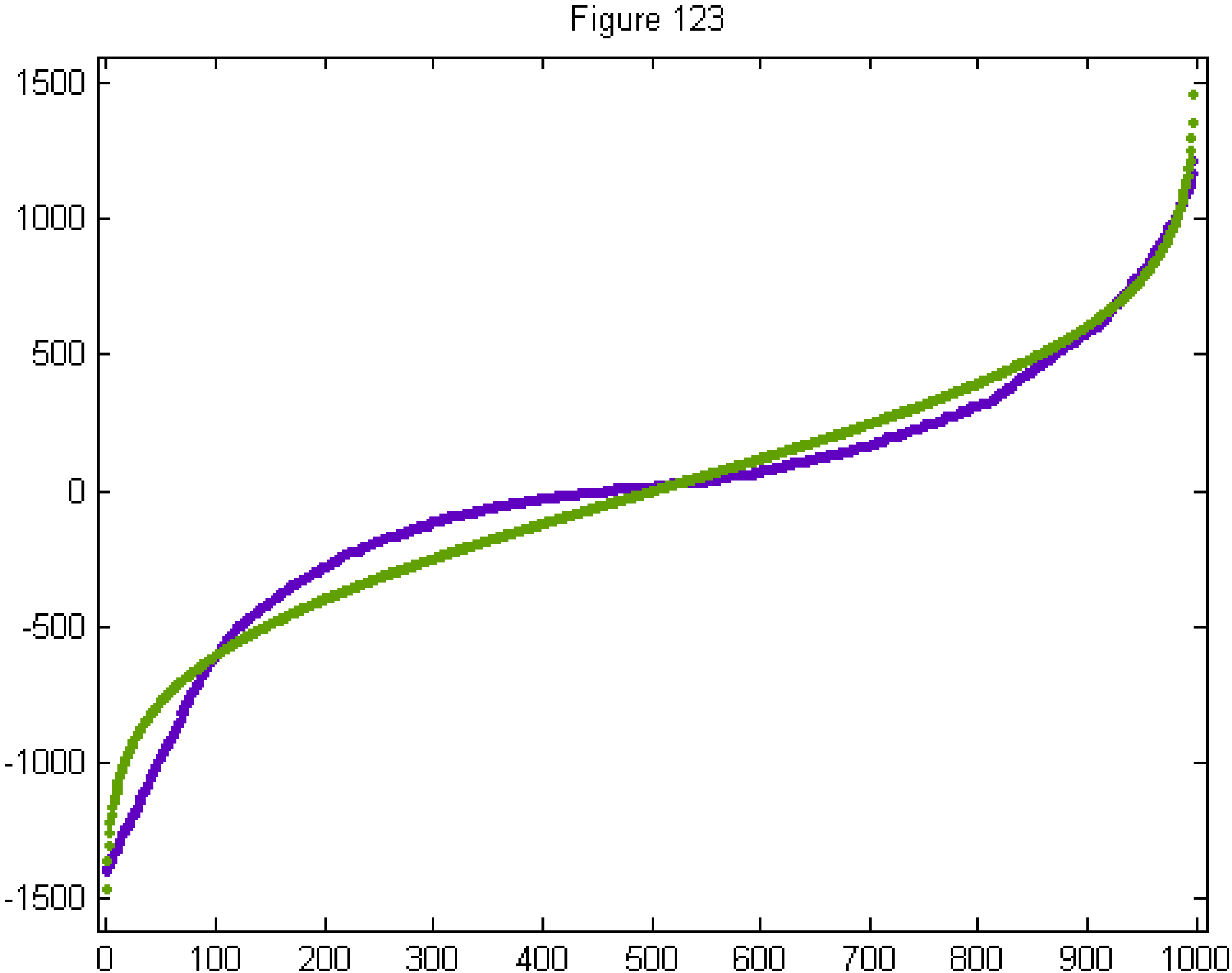


Figure 123
1500
1000
500
0
-500
-1000
-1500
0
100
200
300
400
500
600
700
800
900
1000

Figure 124

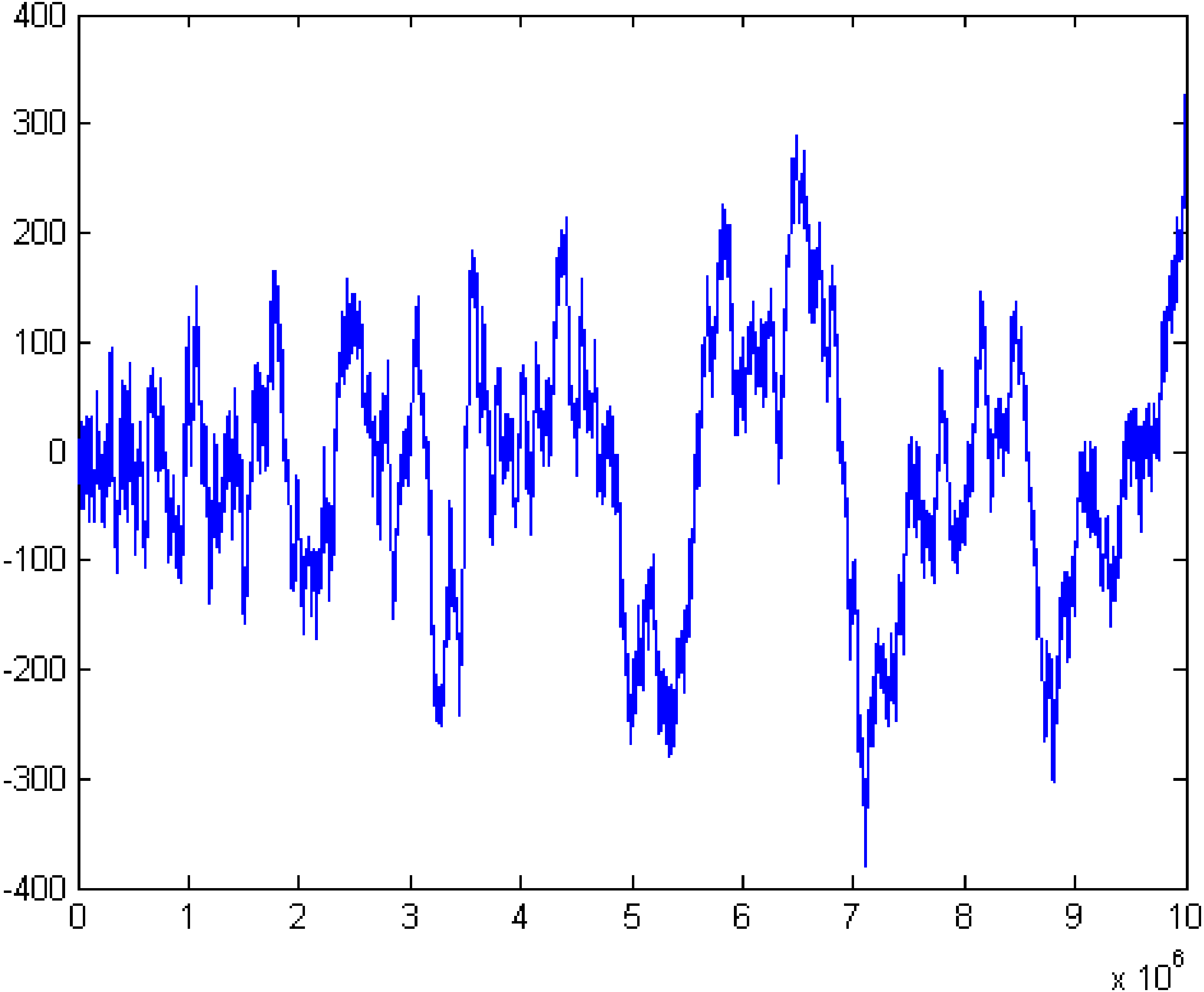


Figure 125

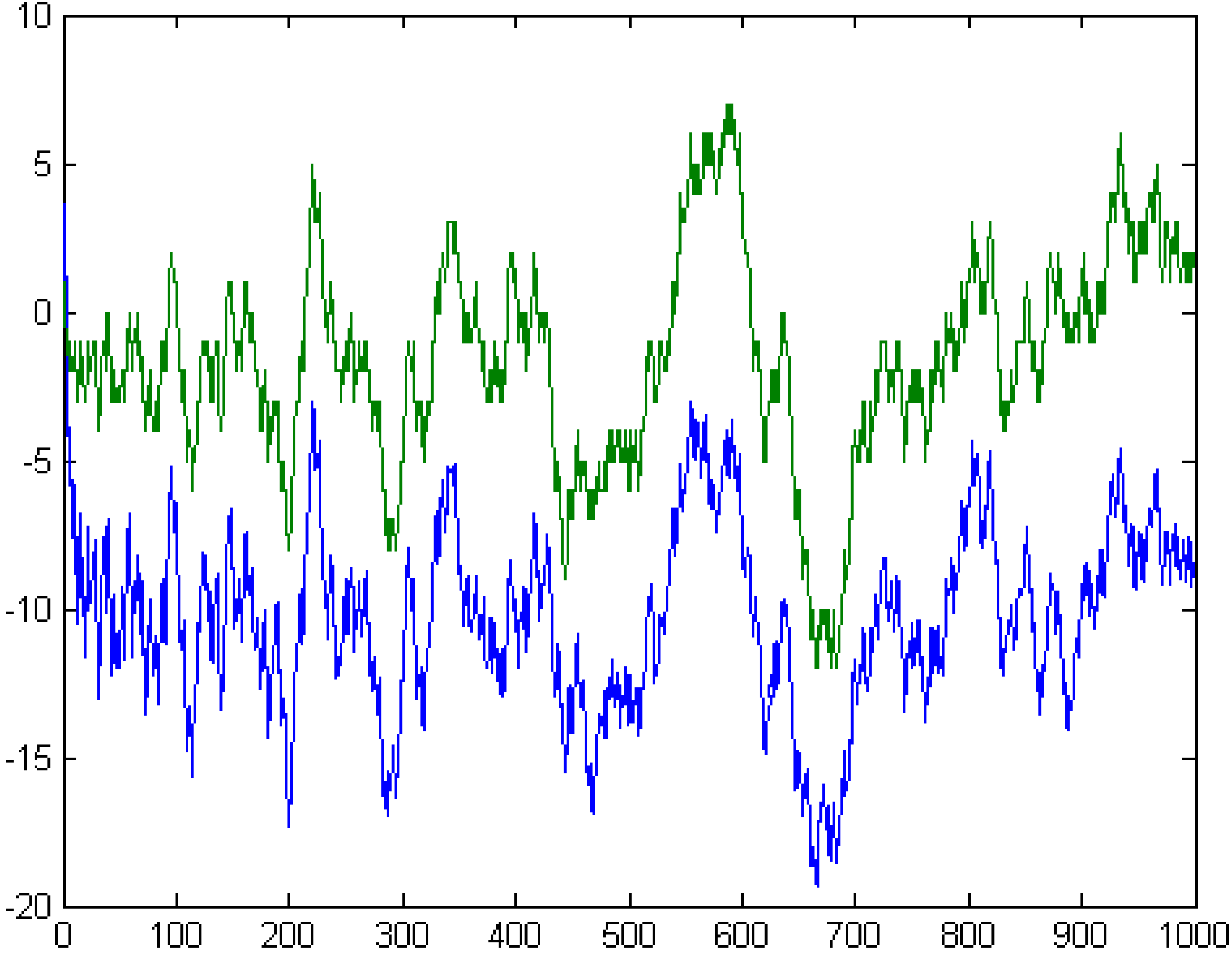
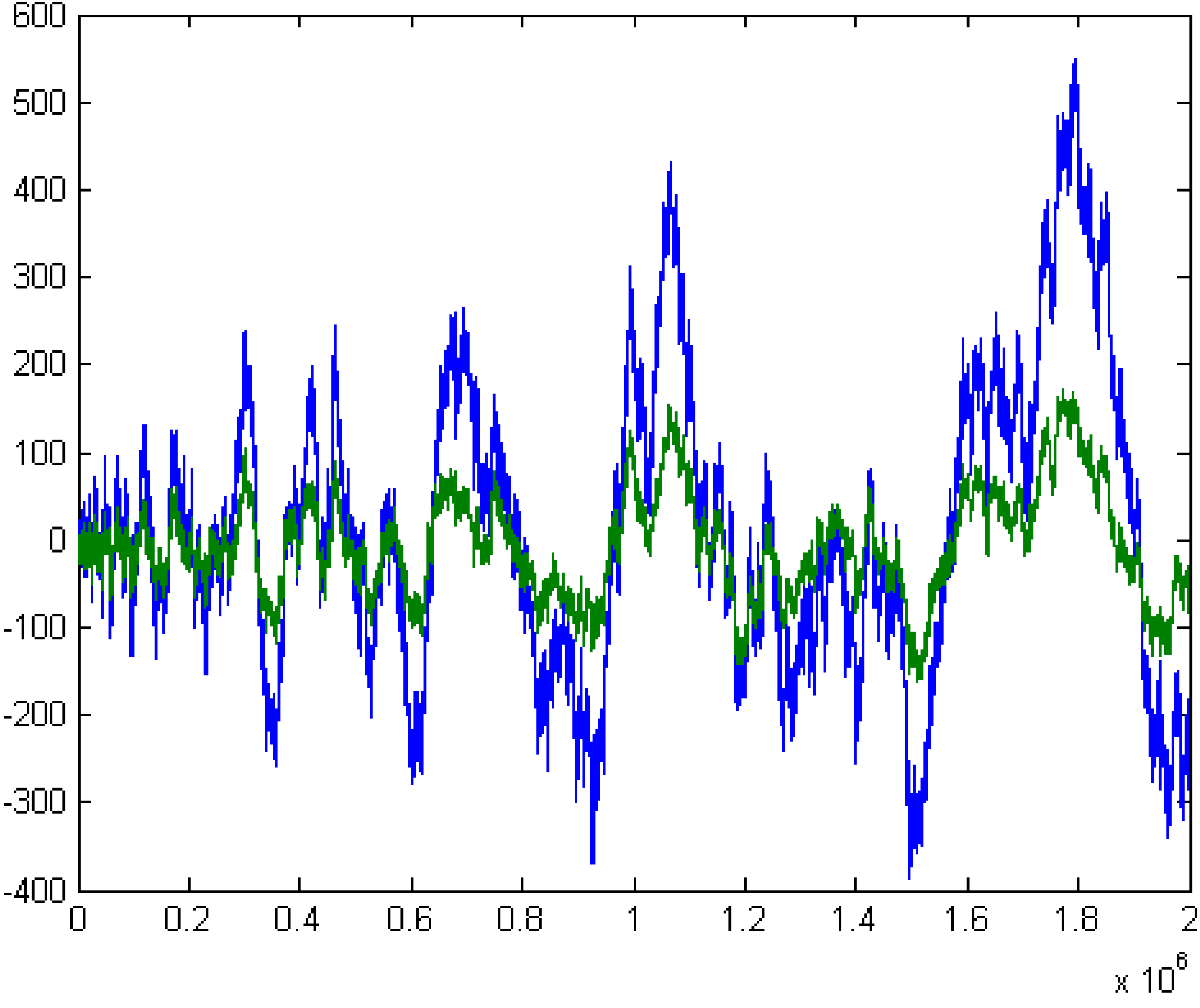


Figure 126



600
500
400
300
200
100
0
-100
-200
-300
-400
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2
x 10^6

Figure 127

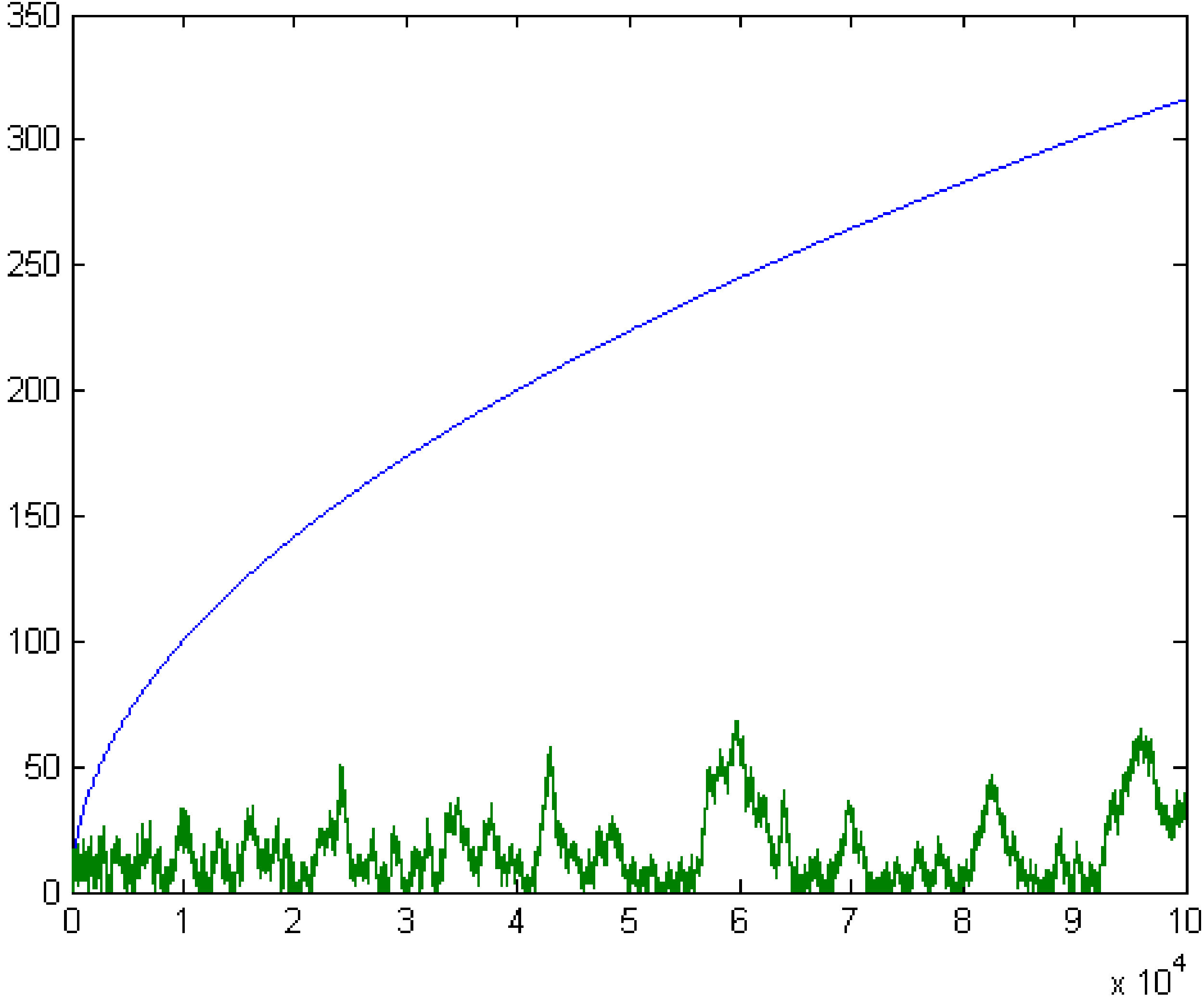
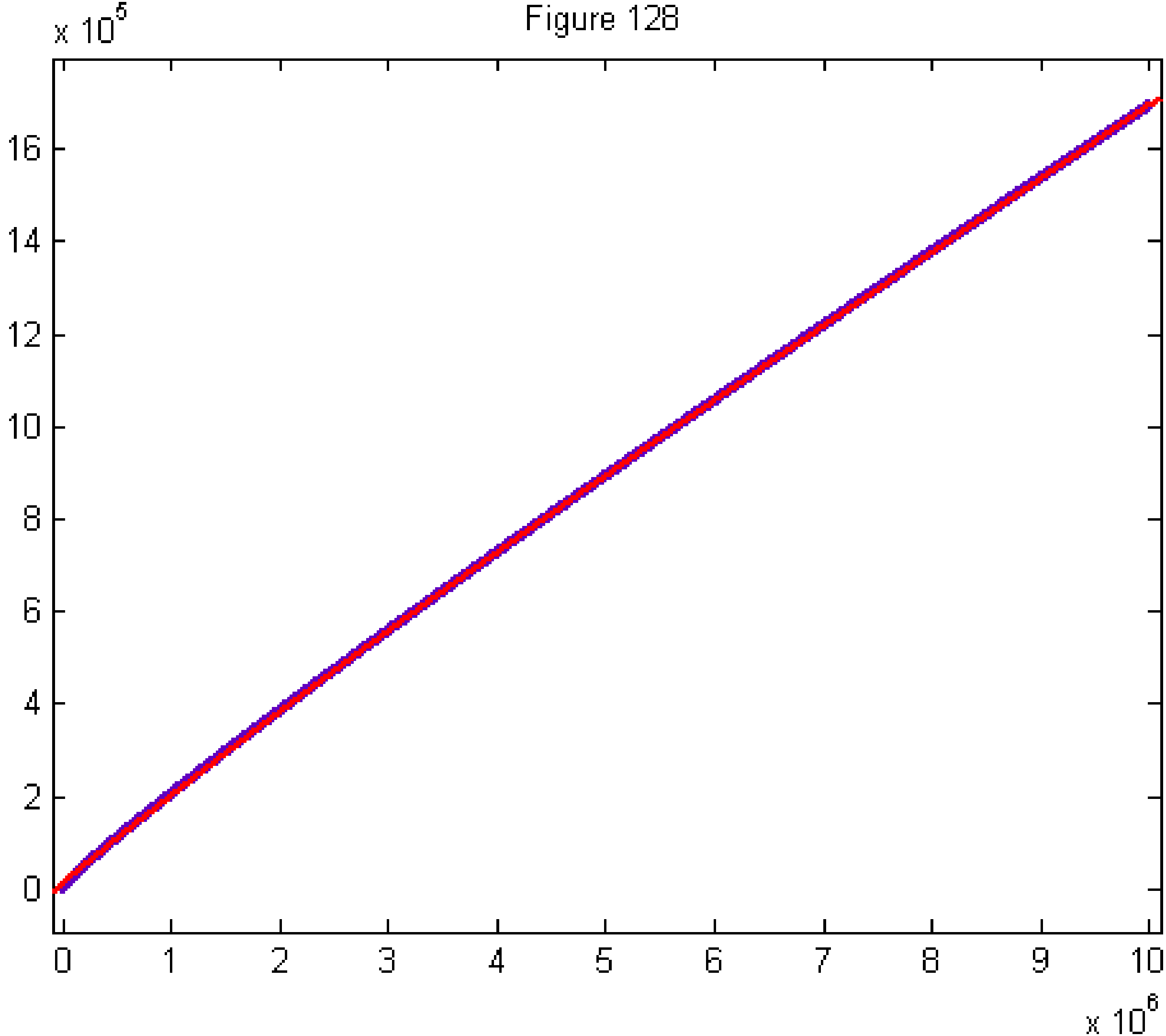


Figure 128



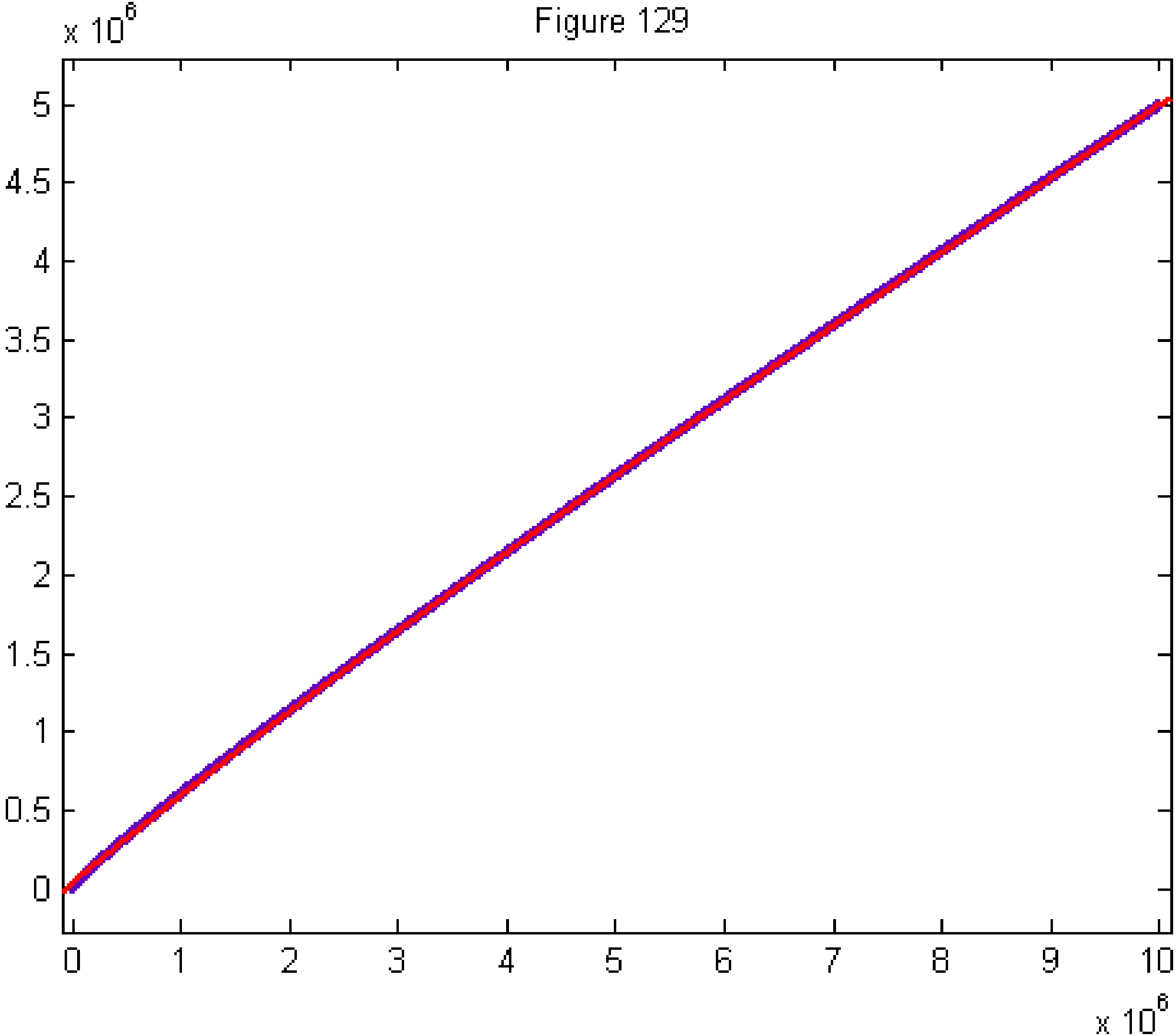


Figure 129
x 10^6
5
4.5
4
3.5
3
2.5
2
1.5
1
0.5
0
0 1 2 3 4 5 6 7 8 9 10
x 10^6

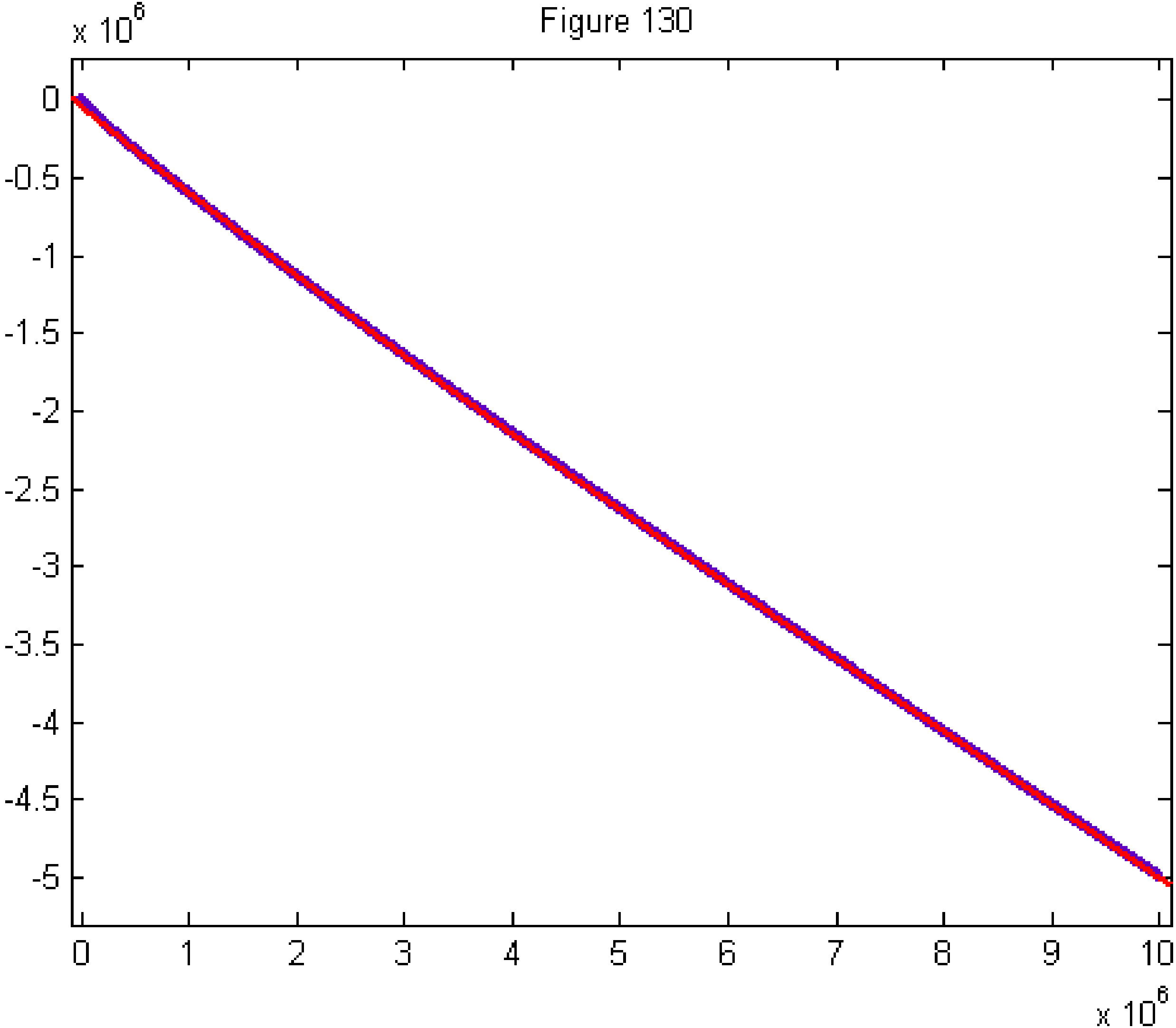
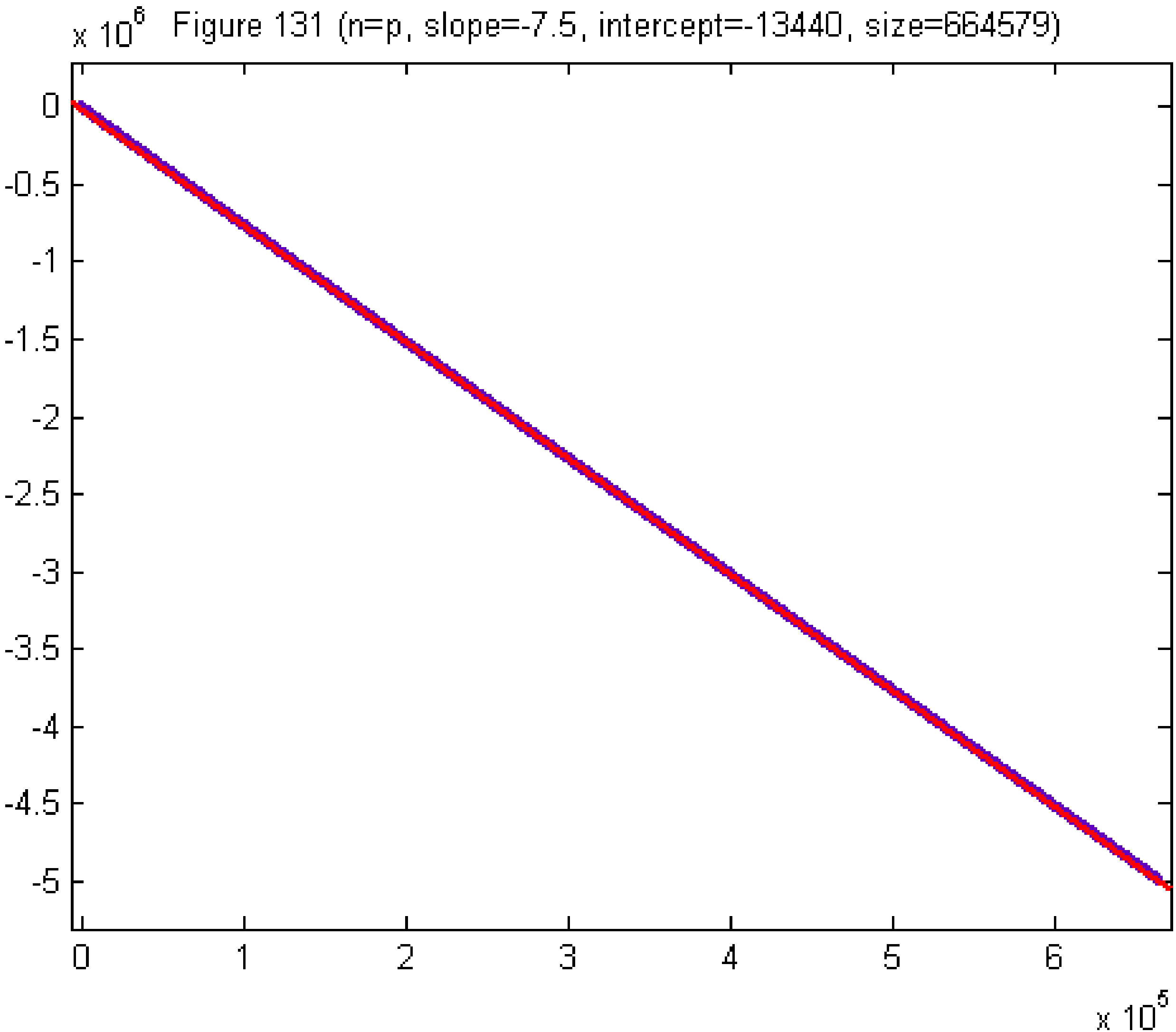
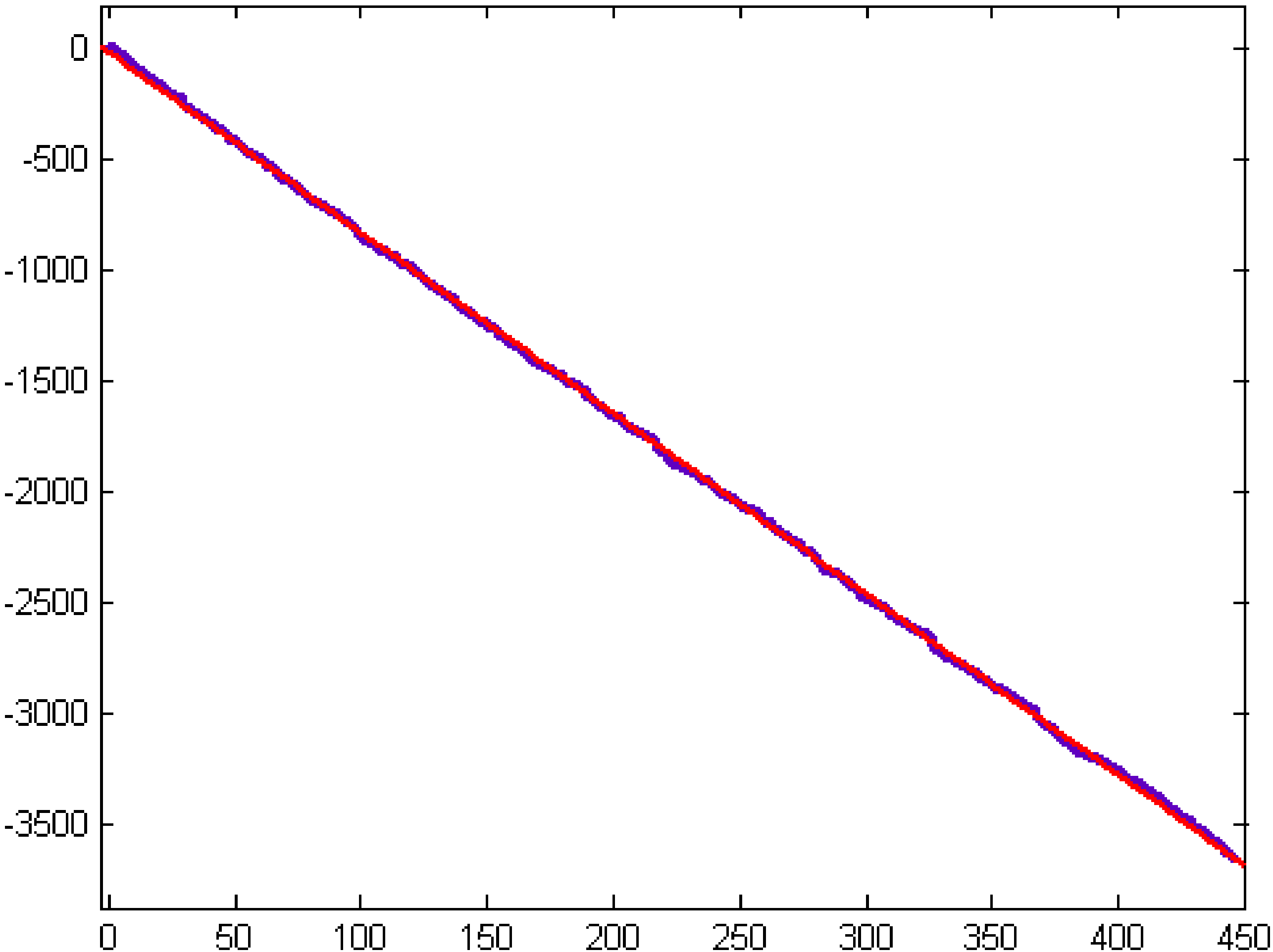


Figure 130
x 10^6
0
-0.5
-1
-1.5
-2
-2.5
-3
-3.5
-4
-4.5
-5
0
1
2
3
4
5
6
7
8
9
10
x 10^6



x 10^6 Figure 131 (n=p, slope=-7.5, intercept=-13440, size=664579)
0
-0.5
-1
-1.5
-2
-2.5
-3
-3.5
-4
-4.5
-5
0
1
2
3
4
5
6
x 10^5

Figure 132 (n=p^2, slope=-8.146, intercept=-22.09, size=446)



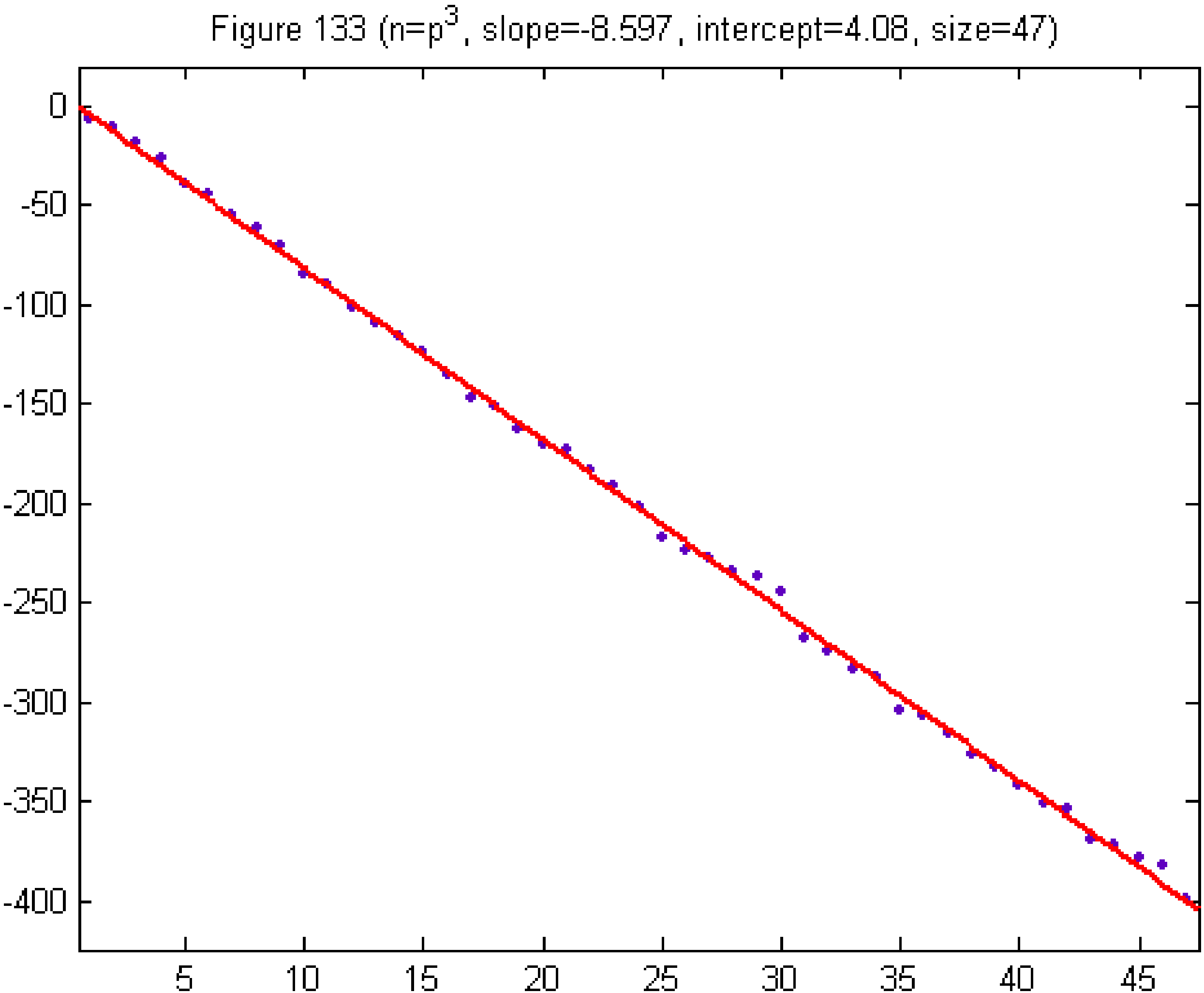


Figure 133 ($n=p^3$, slope=-8.597, intercept=4.08, size=47)
0
-50
-100
-150
-200
-250
-300
-350
-400
5
10
15
20
25
30
35
40
45

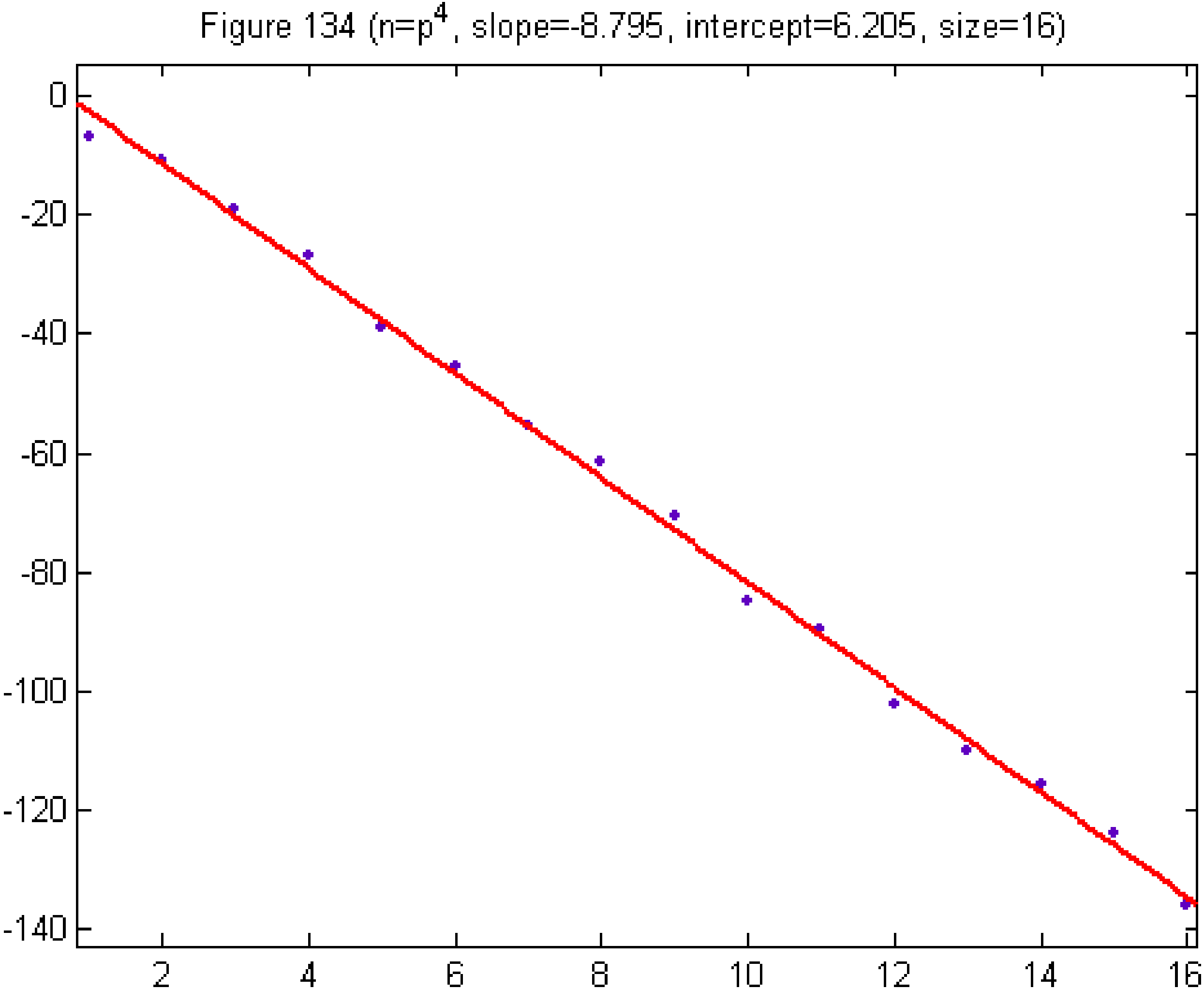


Figure 134 ($n=p^4$, slope=-8.795, intercept=6.205, size=16)
0
-20
-40
-60
-80
-100
-120
-140
2
4
6
8
10
12
14
16

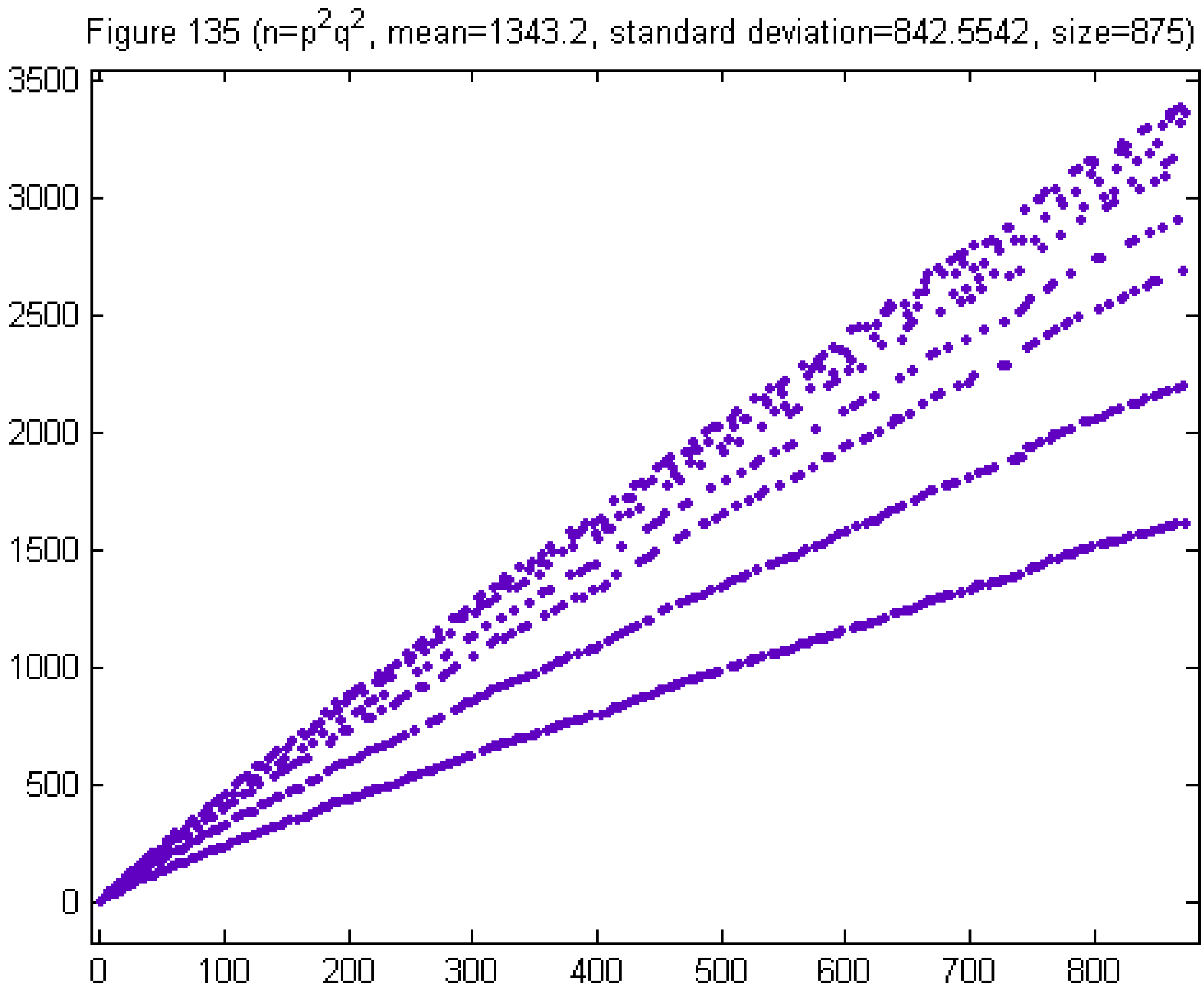
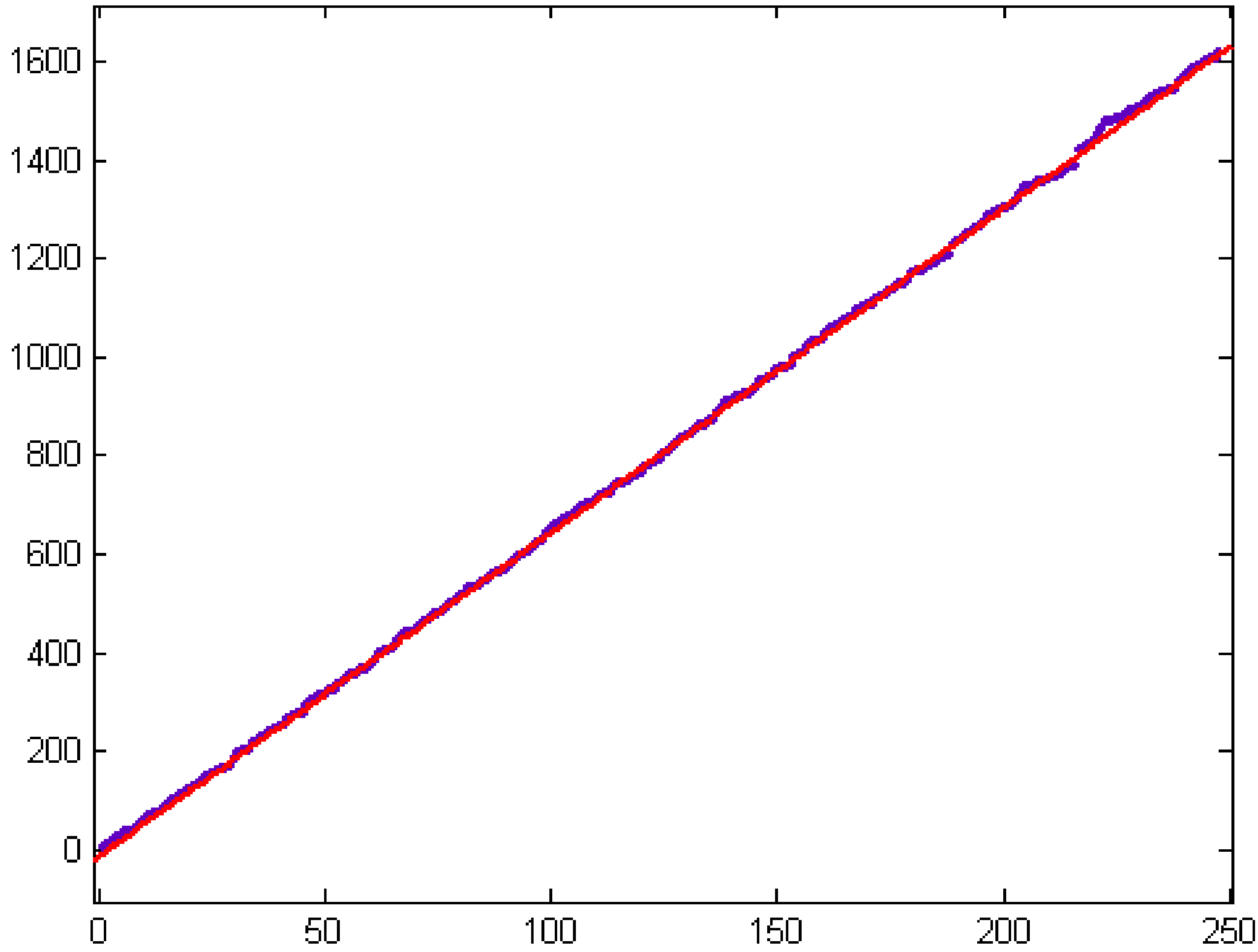


Figure 135 (n=p²q², mean=1343.2, standard deviation=842.5542, size=875)
3500
3000
2500
2000
1500
1000
500
0
0
100
200
300
400
500
600
700
800

Figure 136 (slope=6.57, intercept=-12.9, size=248)



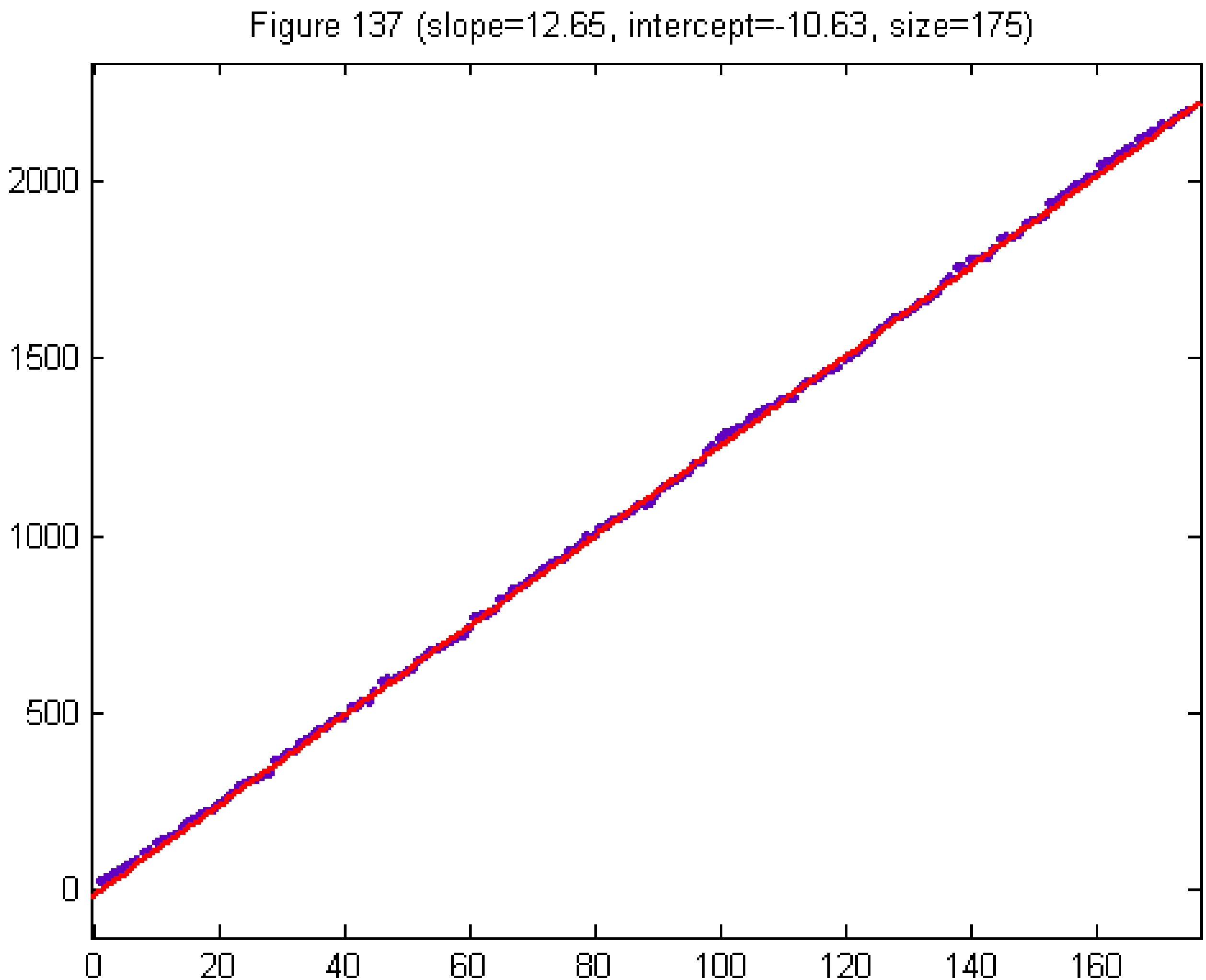


Figure 137 (slope=12.65, intercept=-10.63, size=175)
0
500
1000
1500
2000
0
20
40
60
80
100
120
140
160

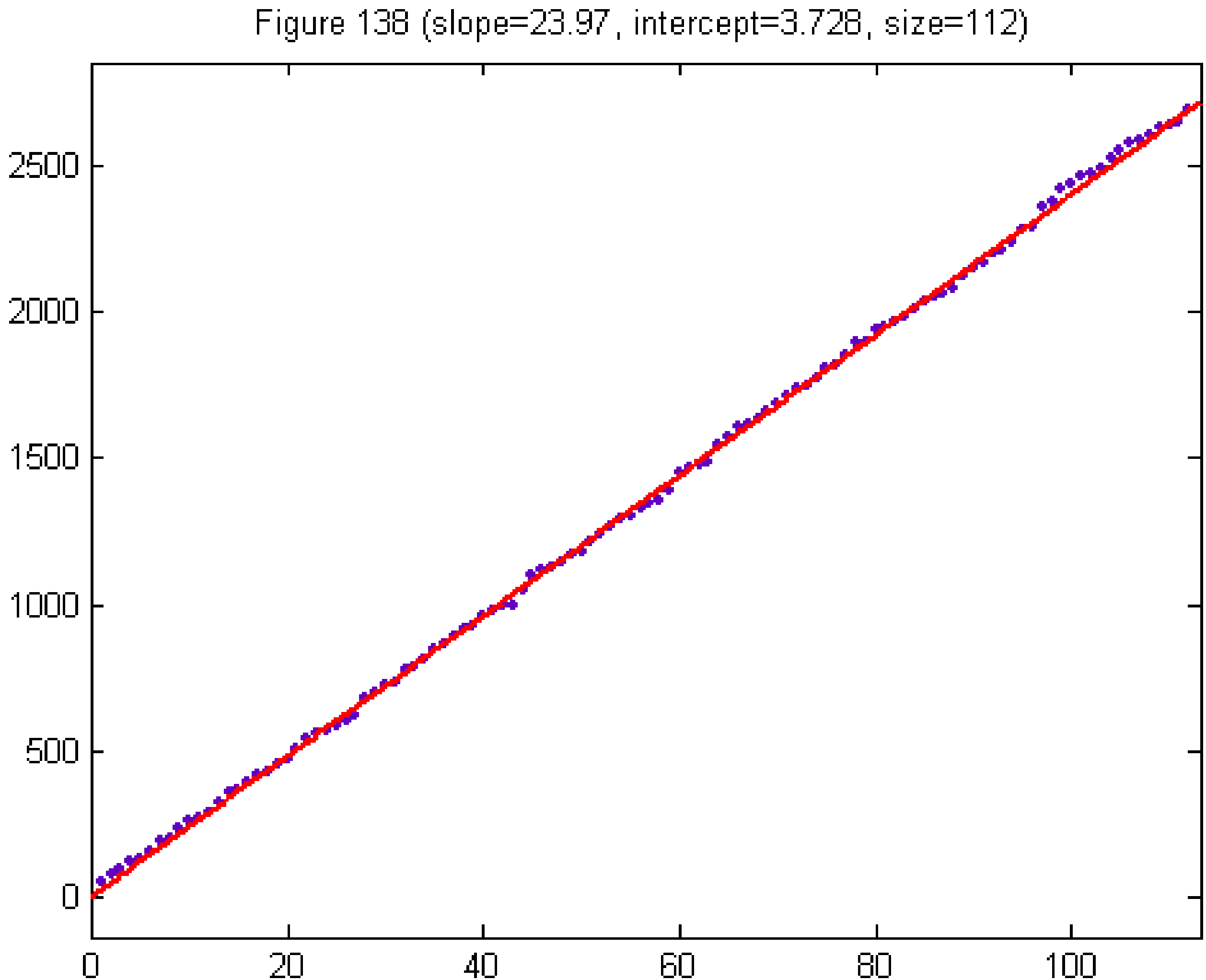


Figure 138 (slope=23.97, intercept=3.728, size=112)
2500
2000
1500
1000
500
0
0
20
40
60
80
100

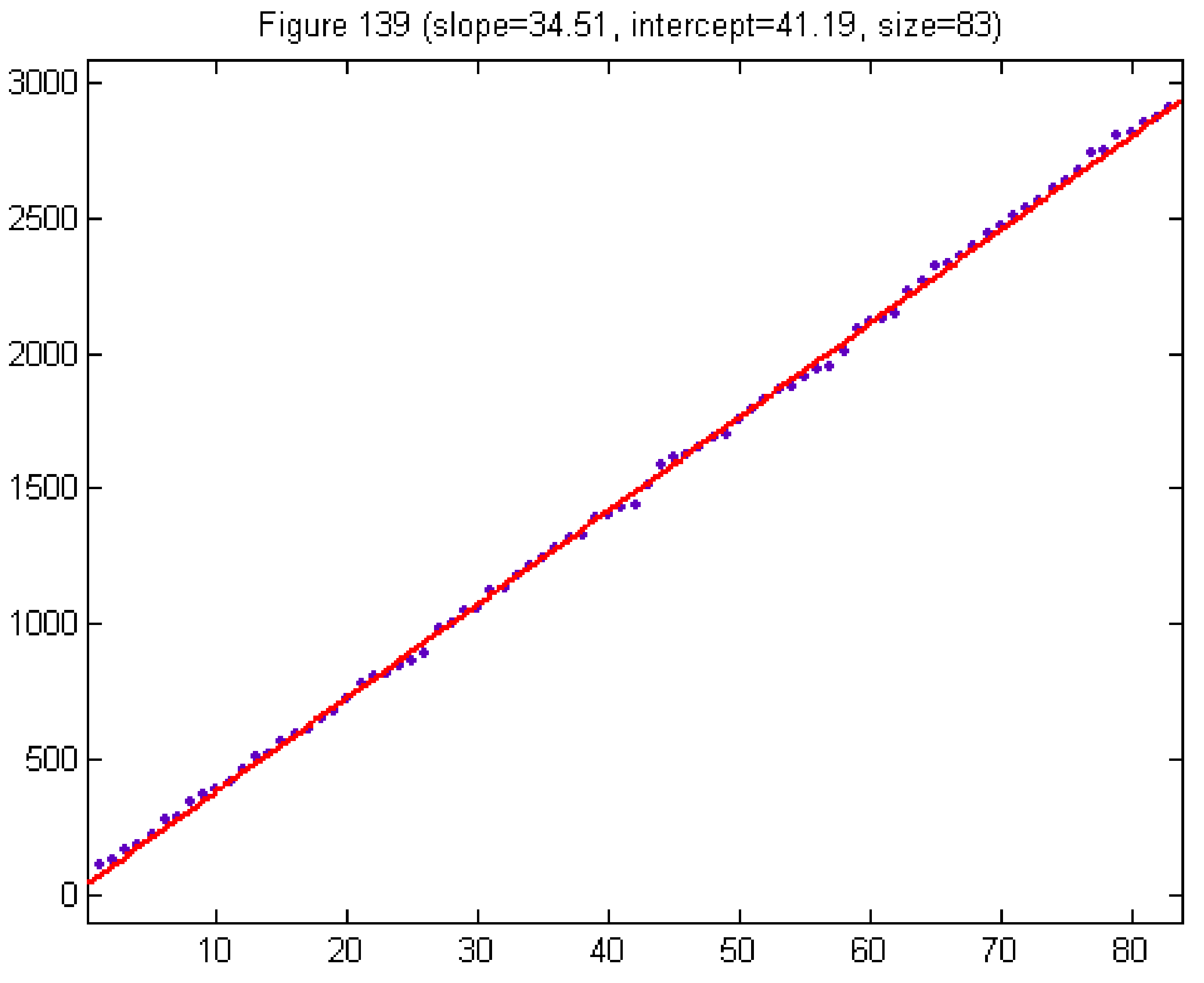


Figure 139 (slope=34.51, intercept=41.19, size=83)
0
500
1000
1500
2000
2500
3000
10
20
30
40
50
60
70
80

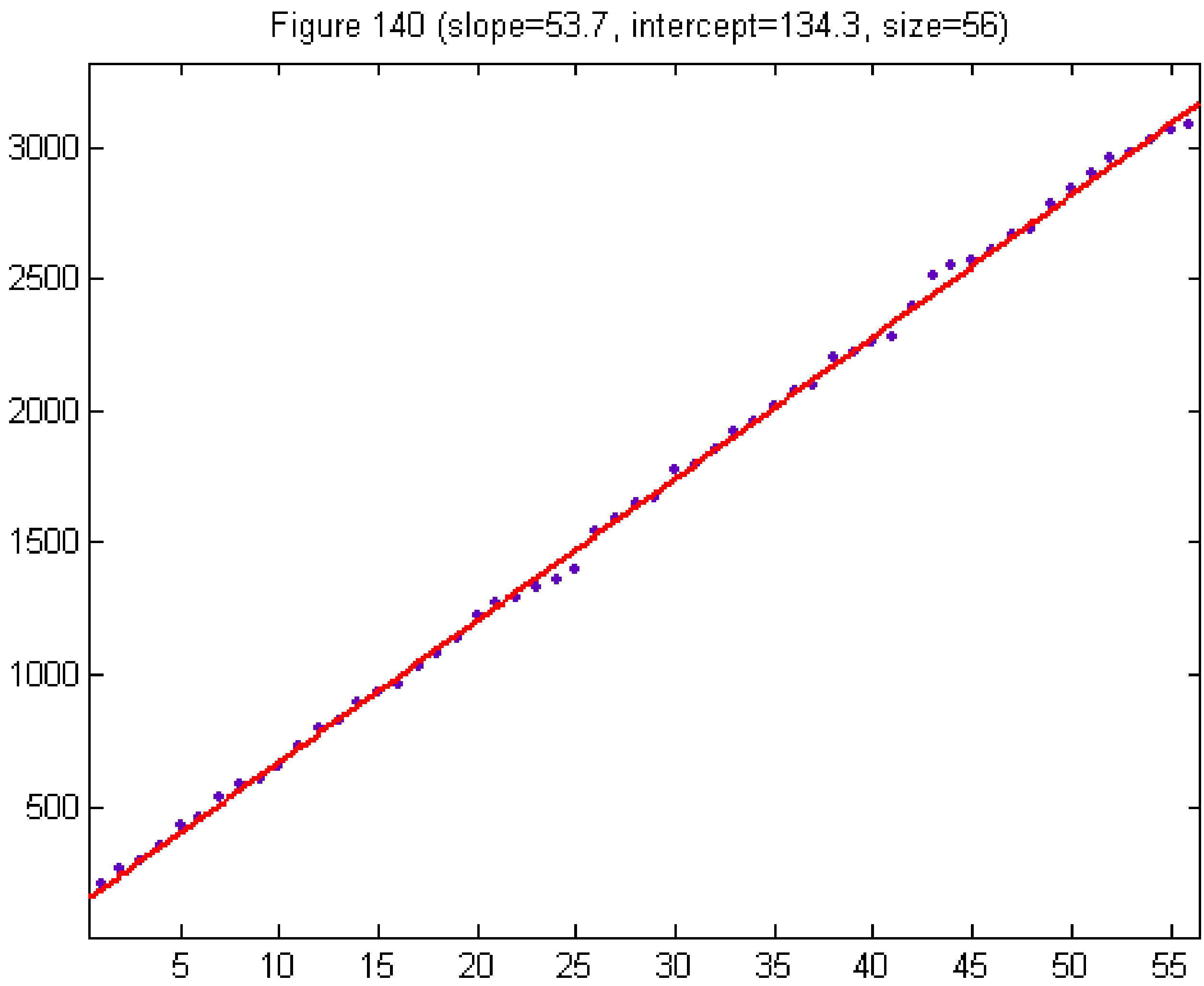
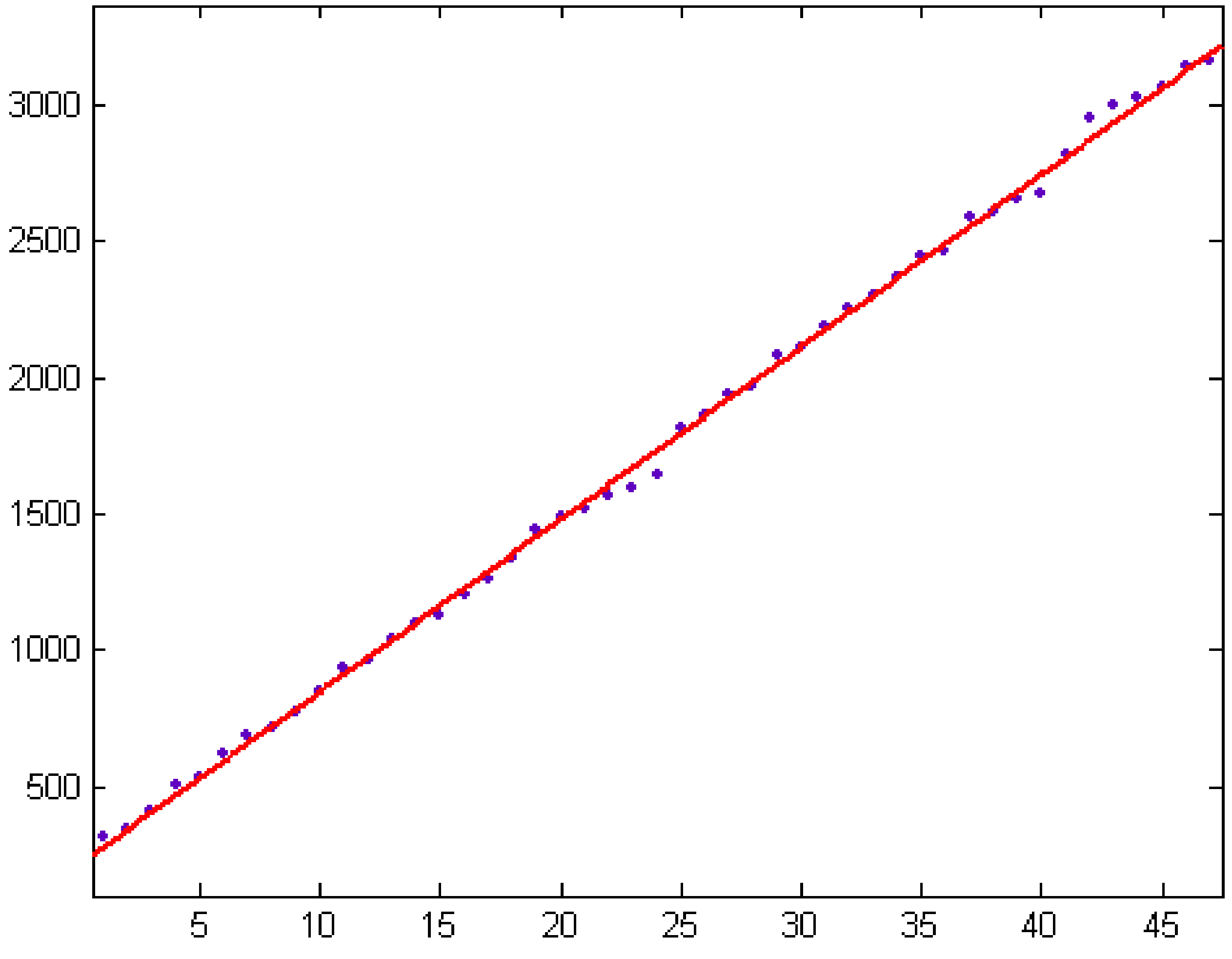


Figure 140 (slope=53.7 , intercept=134.3, size=56)
3000
2500
2000
1500
1000
500
5
10
15
20
25
30
35
40
45
50
55

Figure 141 (slope=63.32, intercept=214.6, size=47)



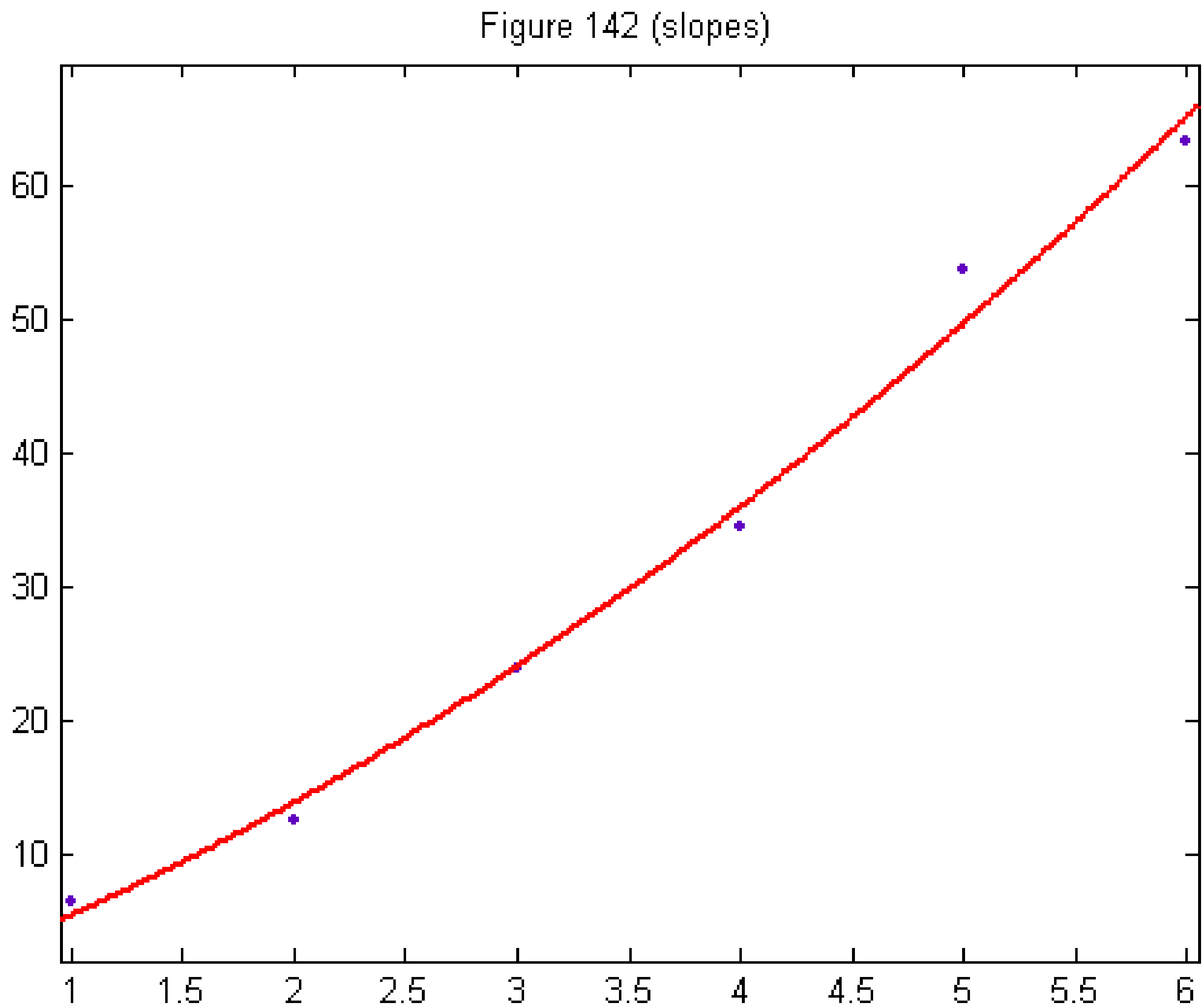
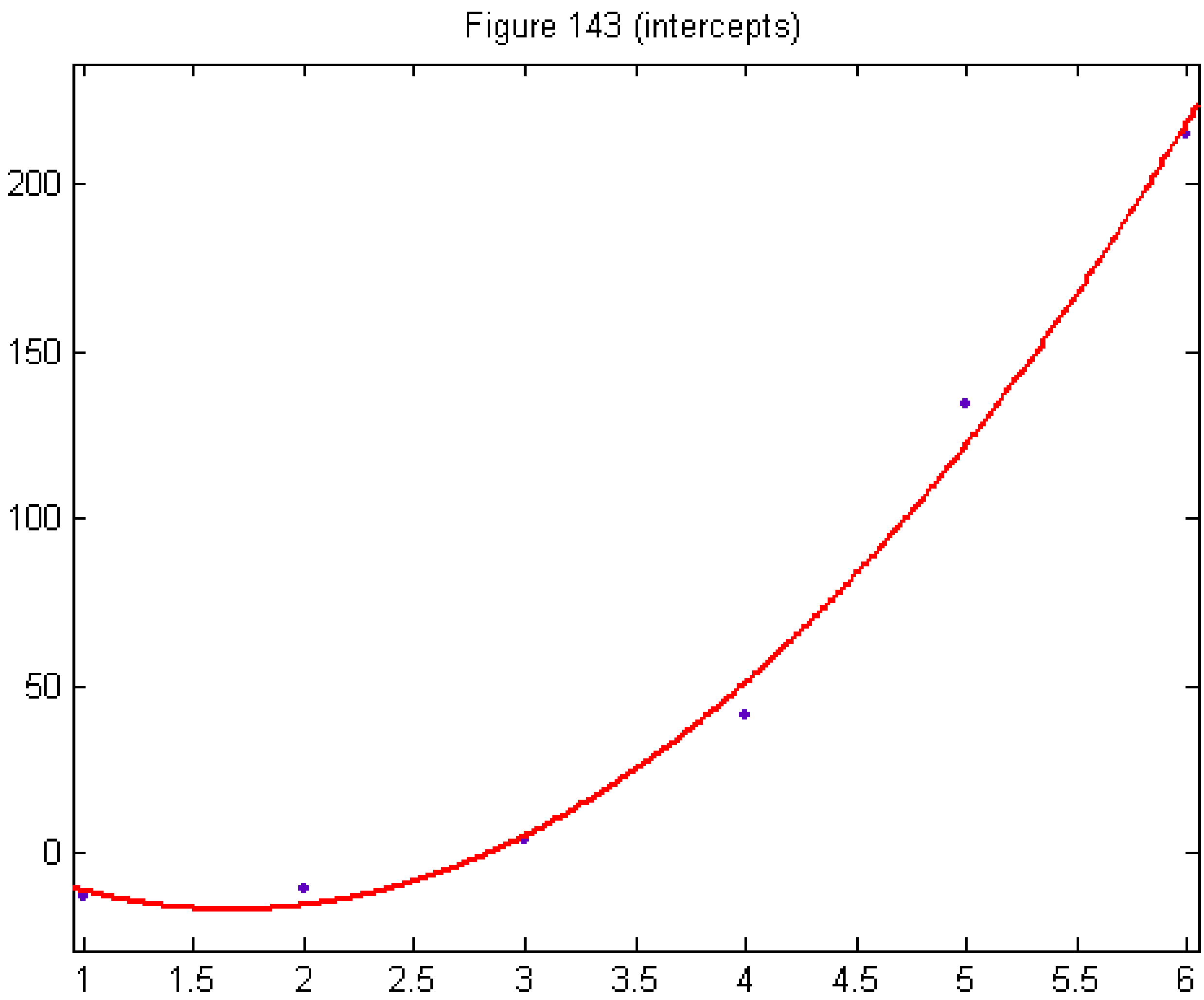
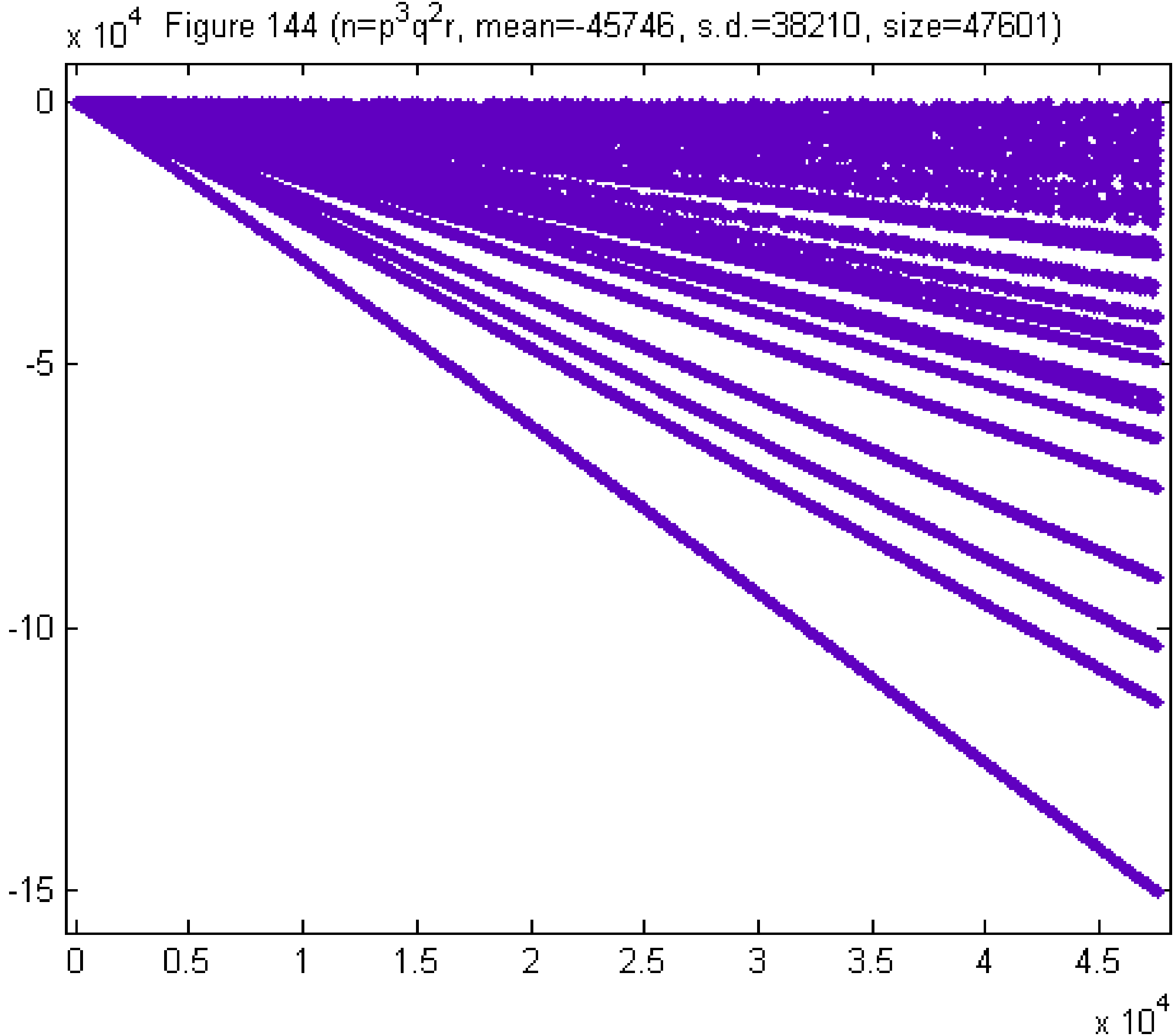


Figure 142 (slopes)
60
50
40
30
20
10
1
1.5
2
2.5
3
3.5
4
4.5
5
5.5
6




x 10^4 Figure 144 (n=p^3q^2r, mean=-45746, s.d.=38210, size=47601)
0
-5
-10
-15
0
0.5
1
1.5
2
2.5
3
3.5
4
4.5
x 10^4

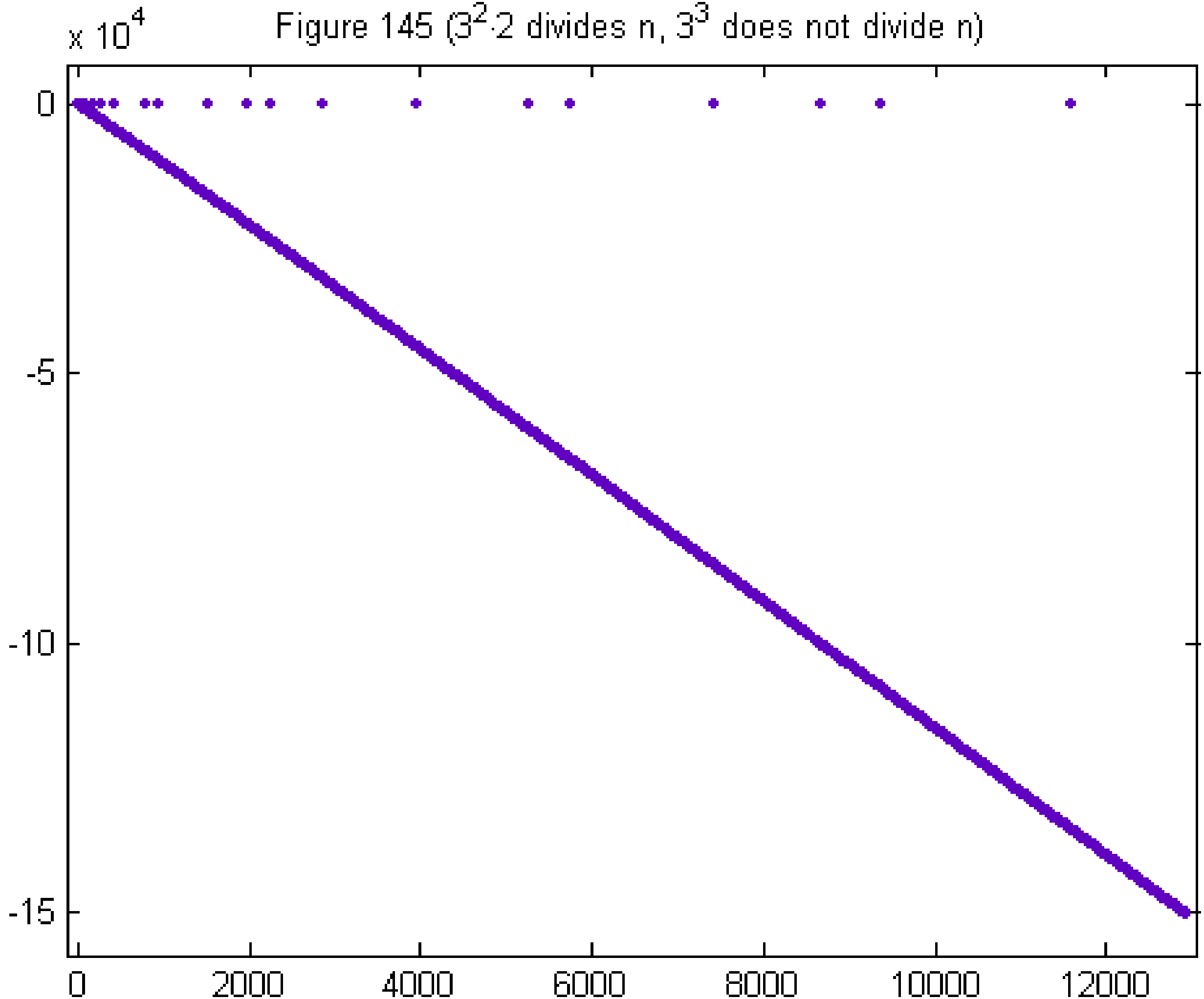


Figure 145 ($3^2 \cdot 2$ divides n, 3^3 does not divide n)
x 10^4
0
-5
-10
-15
0
2000
4000
6000
8000
10000
12000

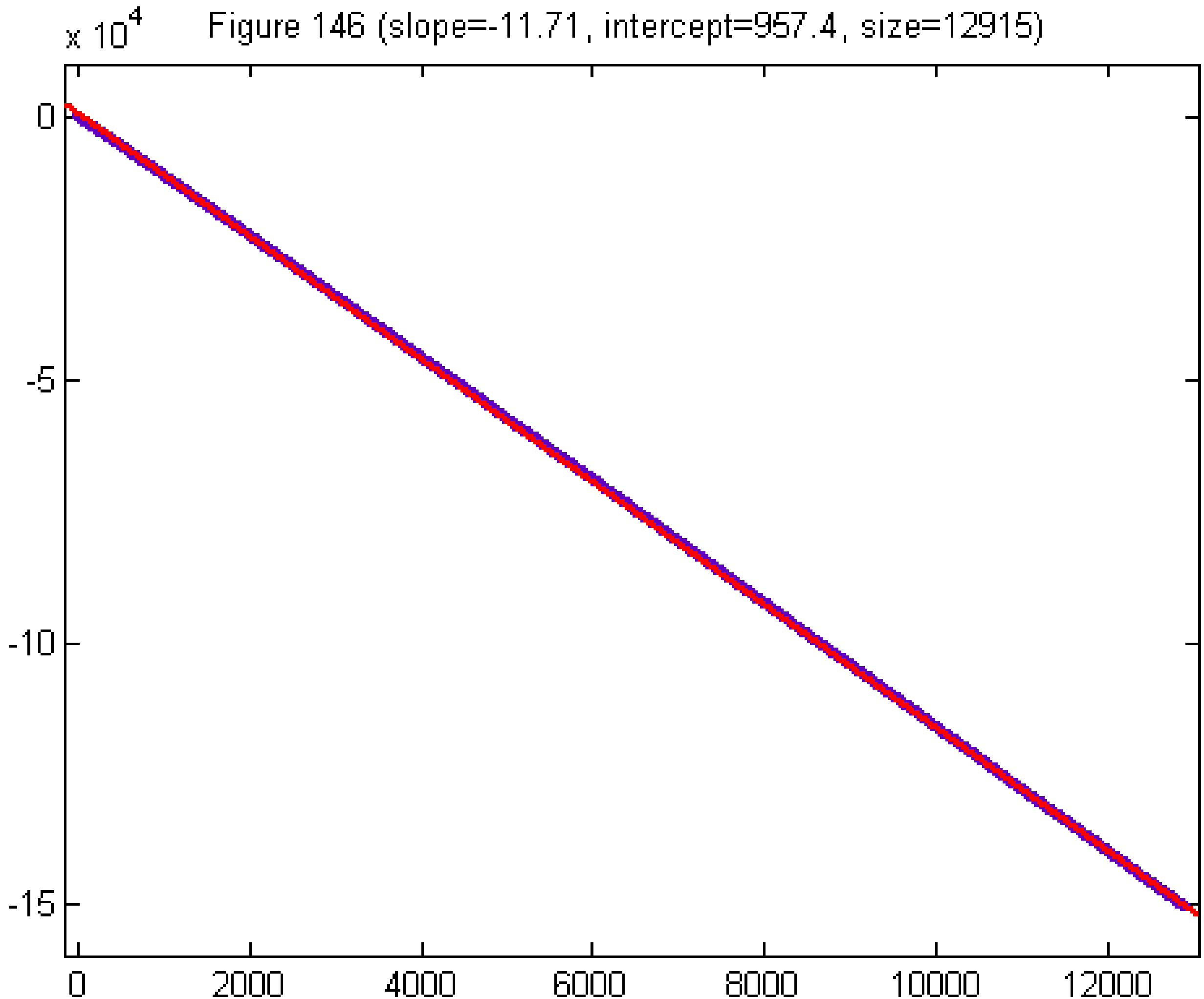
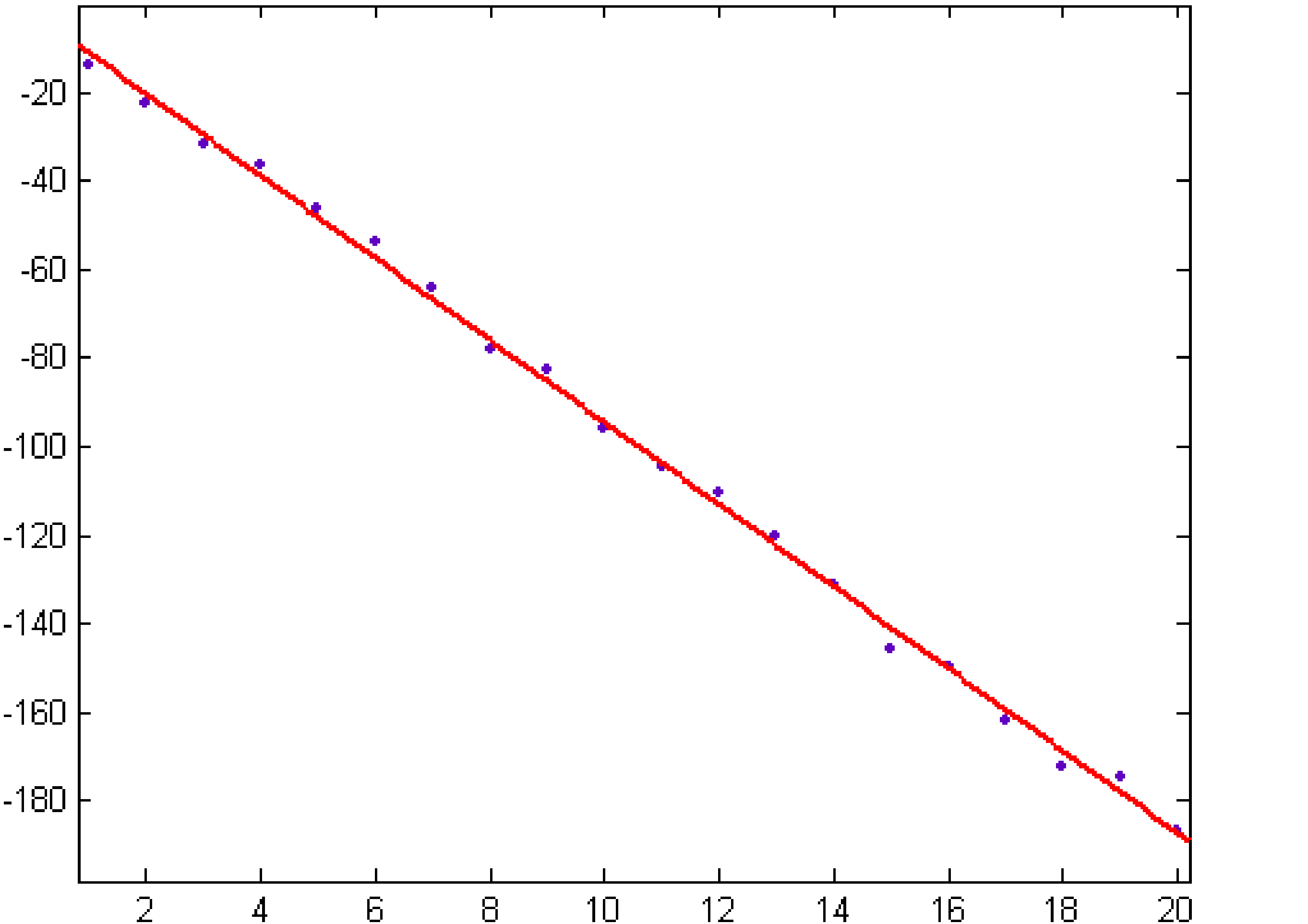


Figure 146 (slope=-11.71, intercept=957.4, size=12915)
x 10^4
0
-5
-10
-15
0
2000
4000
6000
8000
10000
12000

Figure 147 (slope=-9.278, intercept=-1.726, size=20)



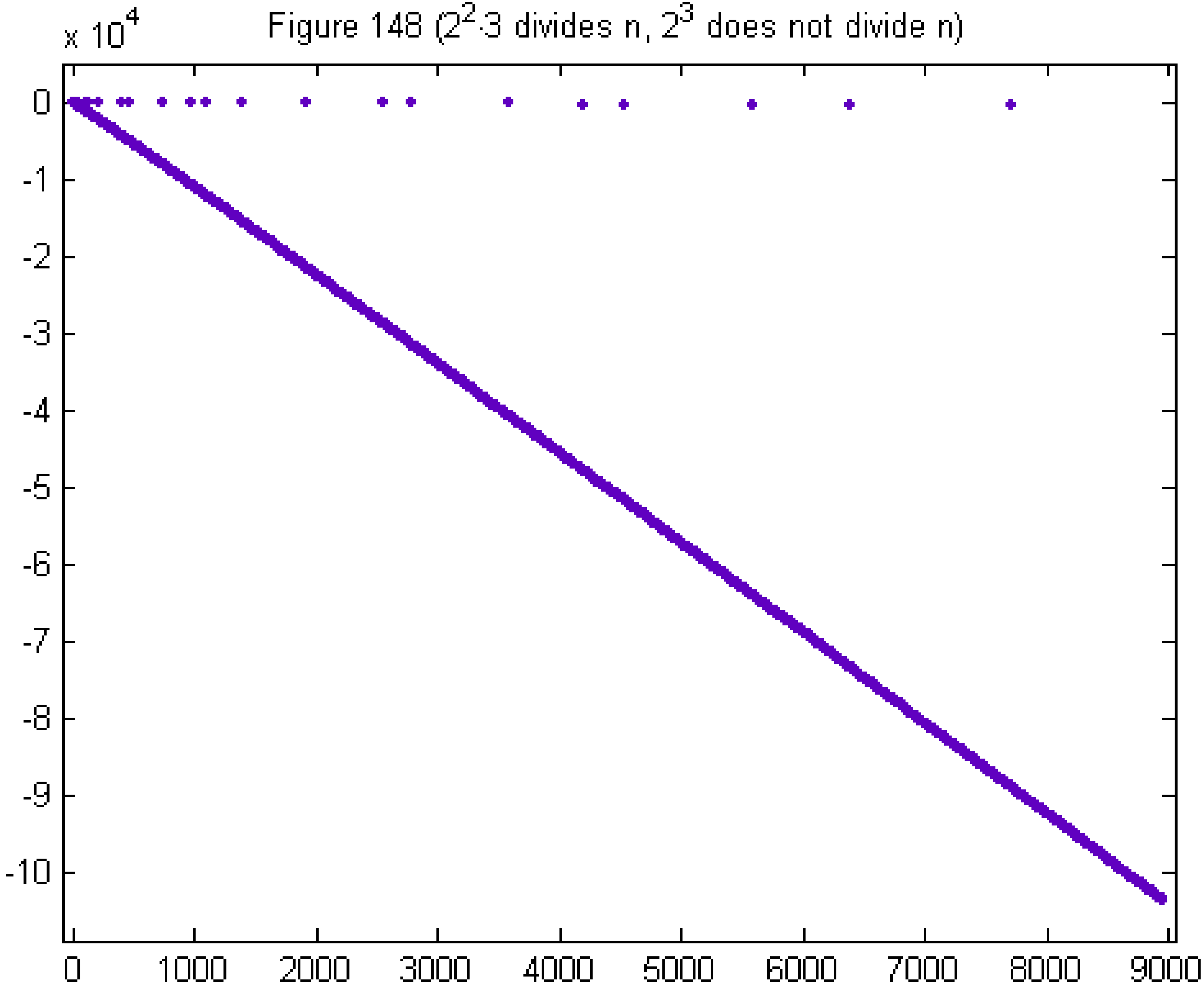


Figure 148 ($2^2 \cdot 3$ divides n, 2^3 does not divide n)
x 10^4
0
-1
-2
-3
-4
-5
-6
-7
-8
-9
-10
0
1000
2000
3000
4000
5000
6000
7000
8000
9000

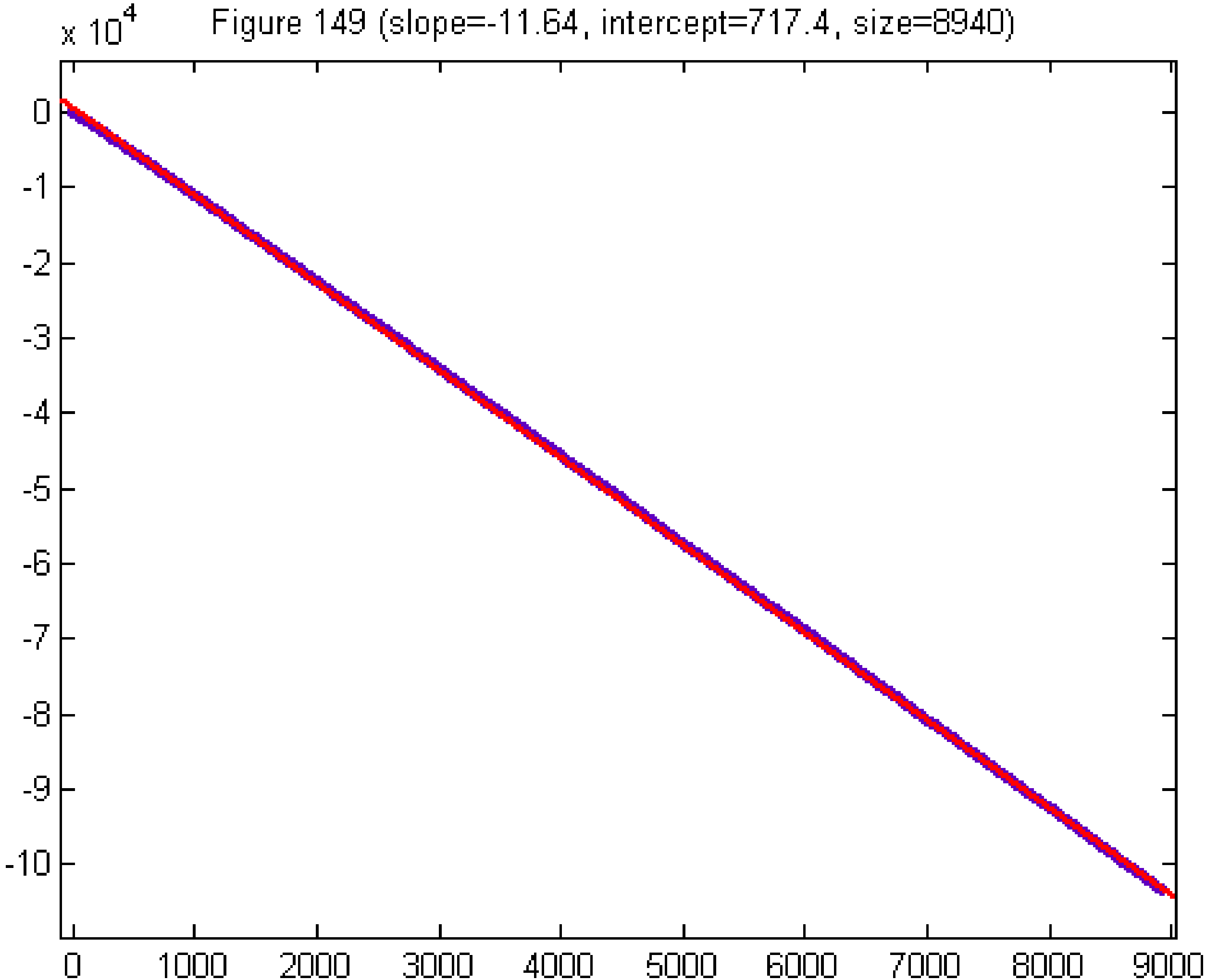


Figure 149 (slope=-11.64, intercept=717.4, size=8940)
x 10^4
0
-1
-2
-3
-4
-5
-6
-7
-8
-9
-10
0
1000
2000
3000
4000
5000
6000
7000
8000
9000

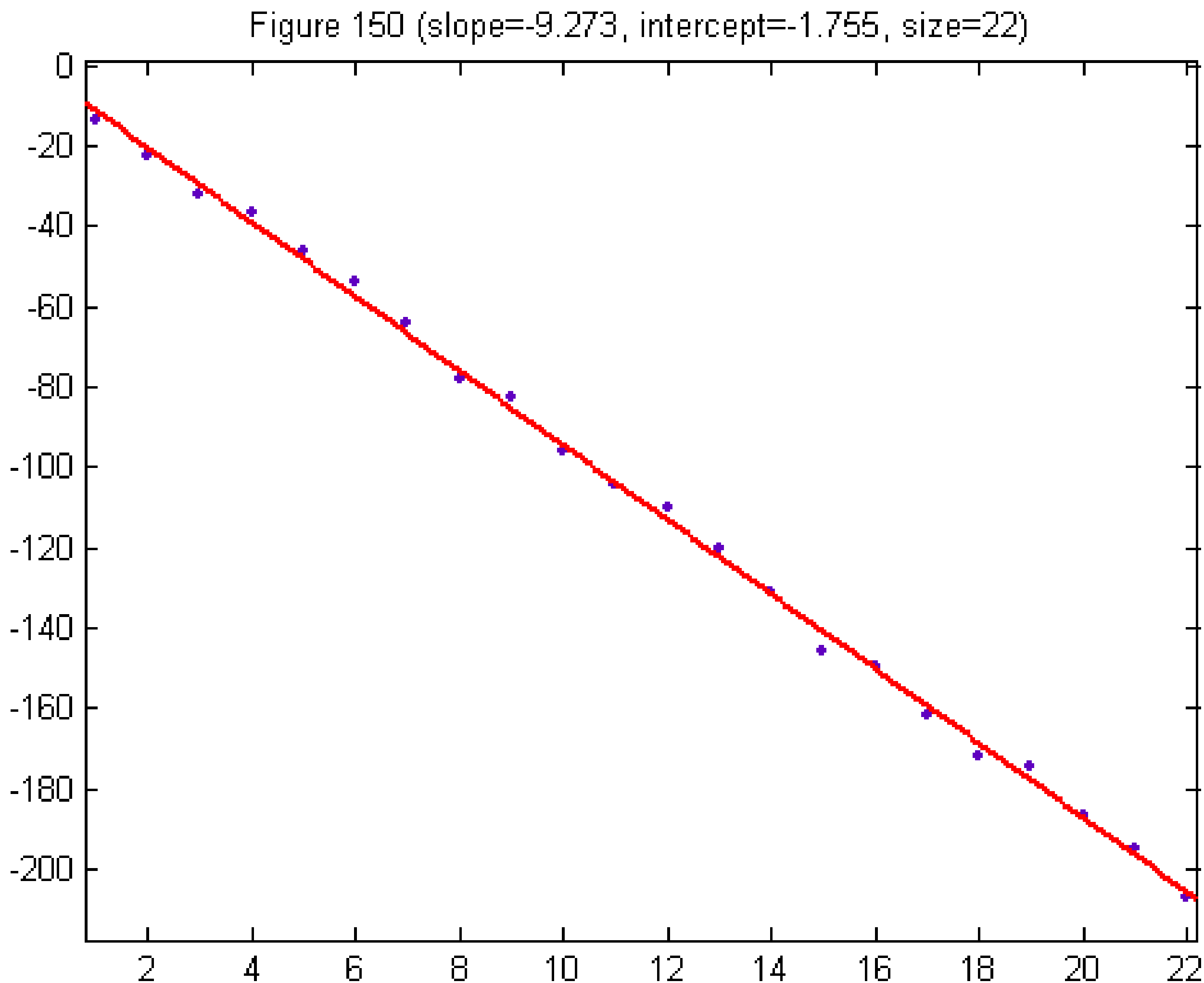


Figure 150 (slope=-9.273, intercept=-1.755, size=22)
0
-20
-40
-60
-80
-100
-120
-140
-160
-180
-200
2
4
6
8
10
12
14
16
18
20
22

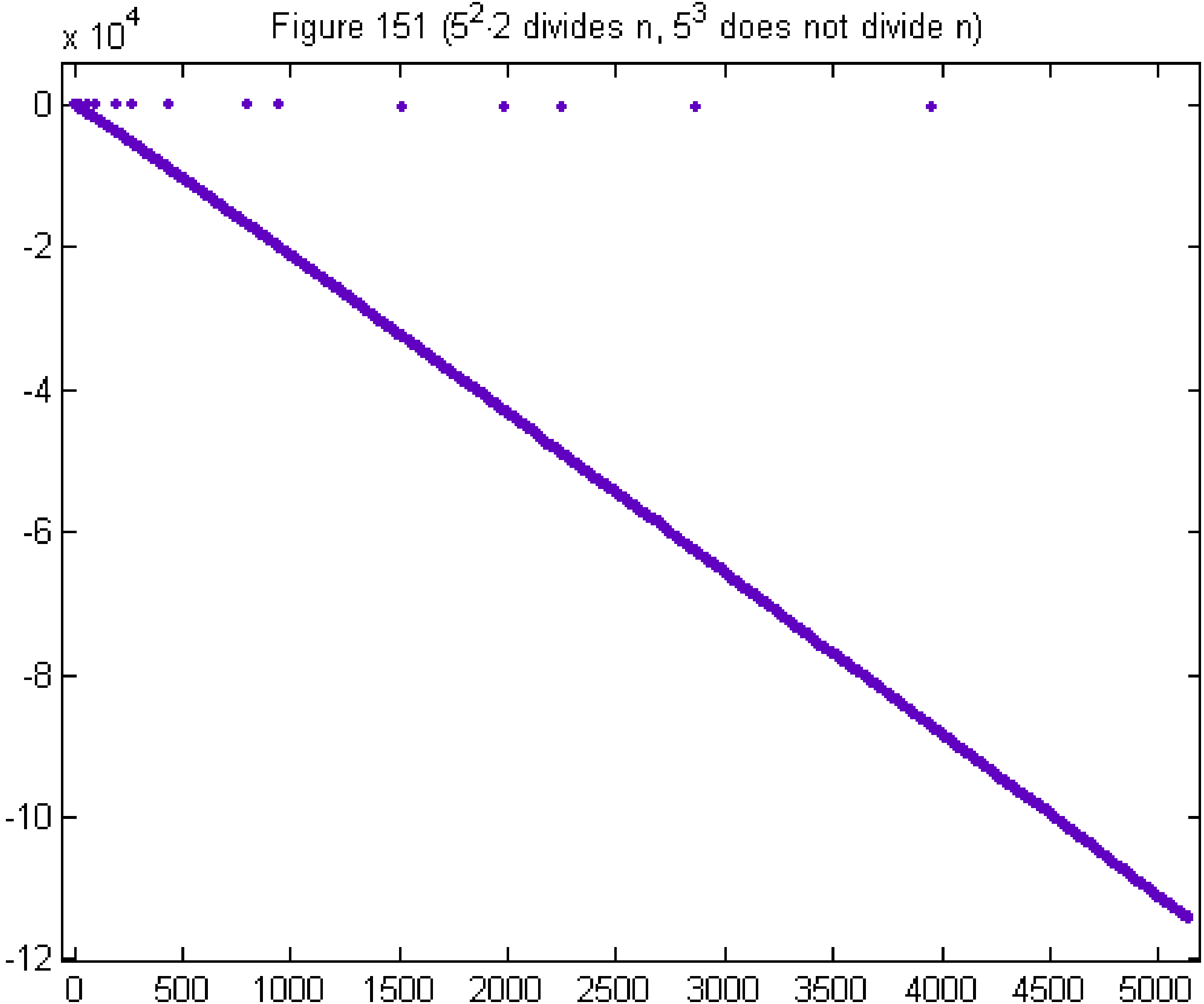
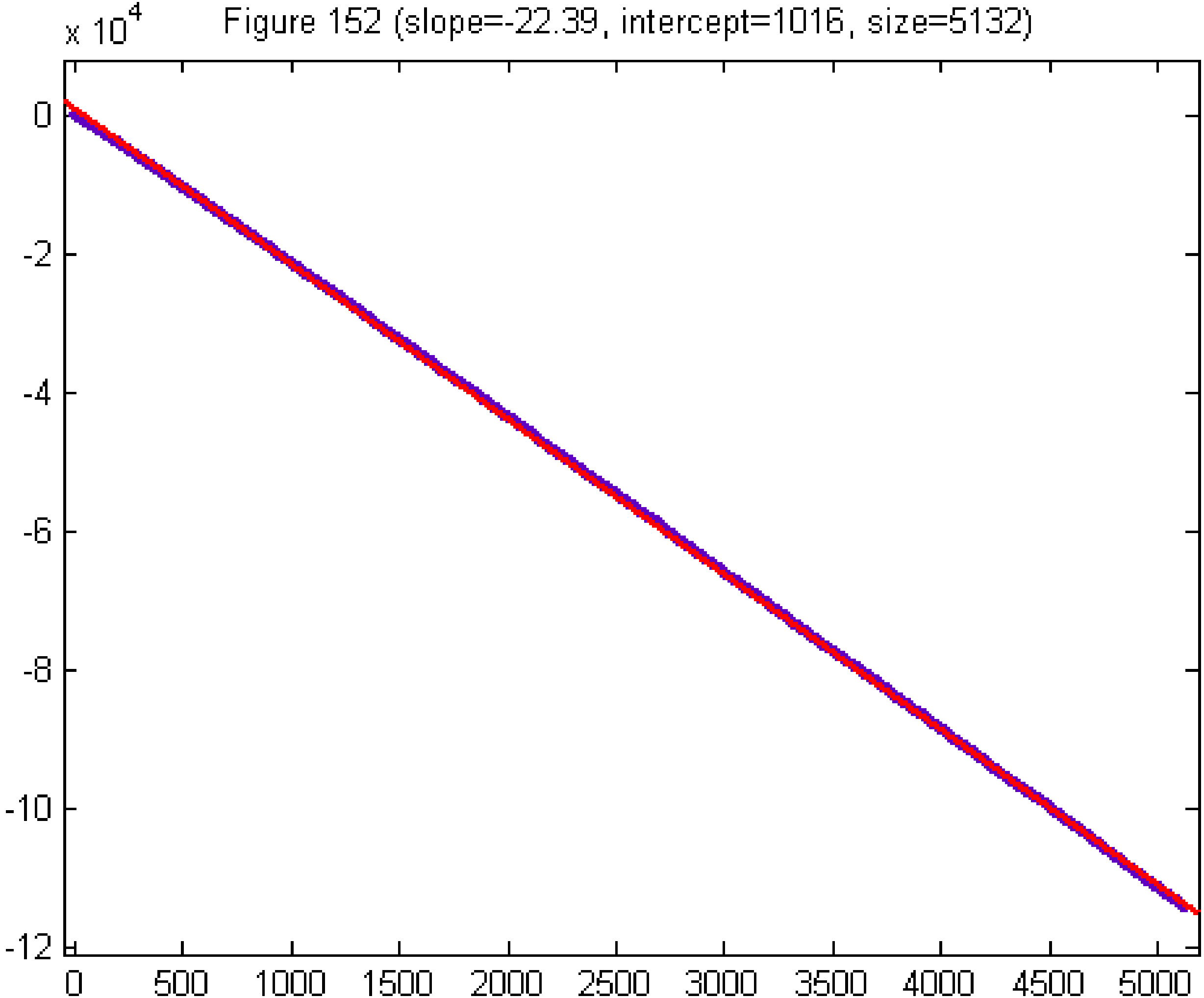


Figure 151 (5^2·2 divides n, 5^3 does not divide n)
x 10^4
0
-2
-4
-6
-8
-10
-12
0
500
1000
1500
2000
2500
3000
3500
4000
4500
5000


x 10^4
Figure 152 (slope=-22.39, intercept=1016, size=5132)
0
-2
-4
-6
-8
-10
-12
0
500
1000
1500
2000
2500
3000
3500
4000
4500
5000

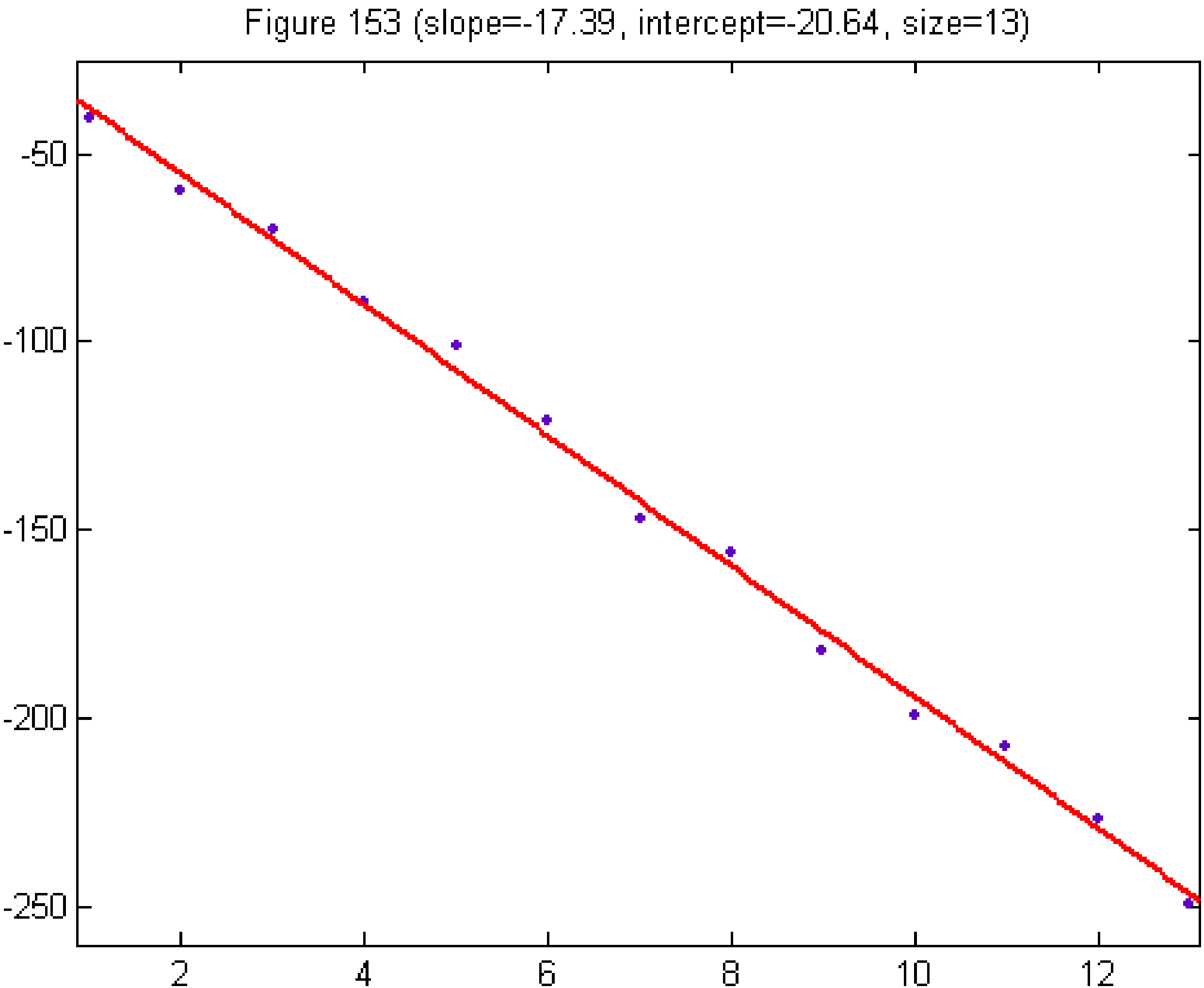


Figure 153 (slope=-17.39, intercept=-20.64, size=13)
-50
-100
-150
-200
-250
2
4
6
8
10
12

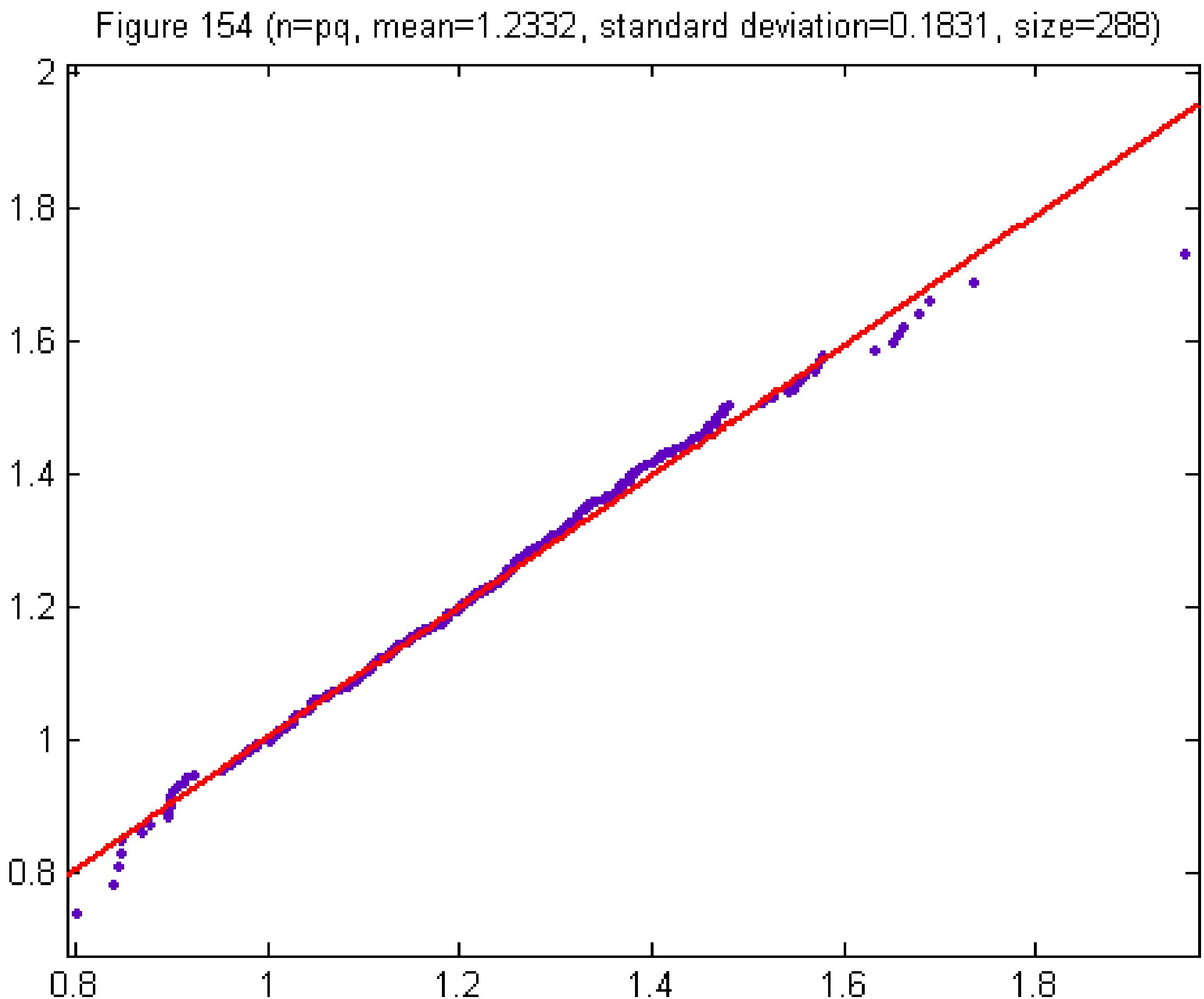


Figure 154 (n=pq, mean=1.2332, standard deviation=0.1831, size=288)
2
1.8
1.6
1.4
1.2
1
0.8
0.8
1
1.2
1.4
1.6
1.8

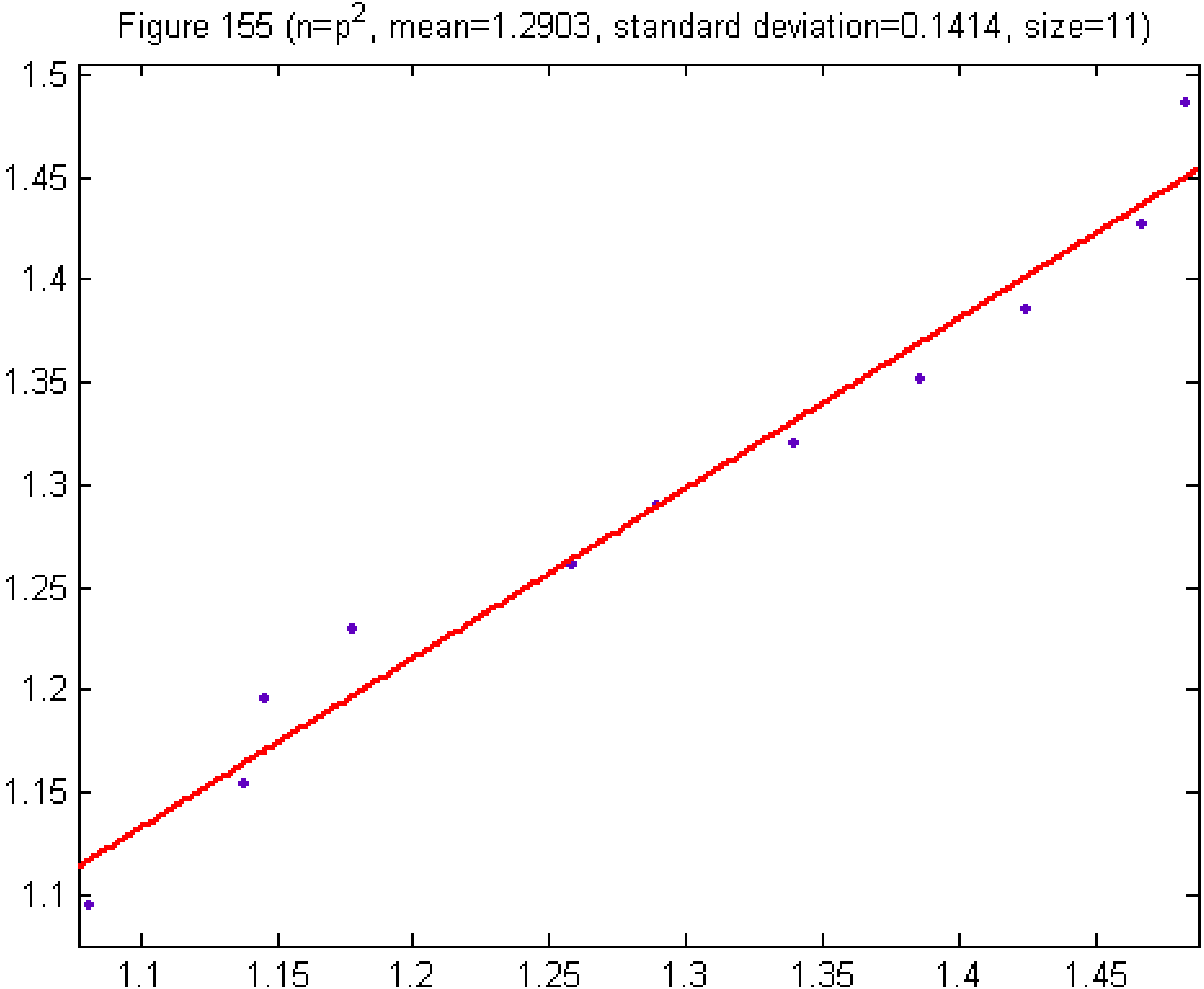


Figure 155 (n=p2, mean=1.2903, standard deviation=0.1414, size=11)
1.5
1.45
1.4
1.35
1.3
1.25
1.2
1.15
1.1
1.1
1.15
1.2
1.25
1.3
1.35
1.4
1.45

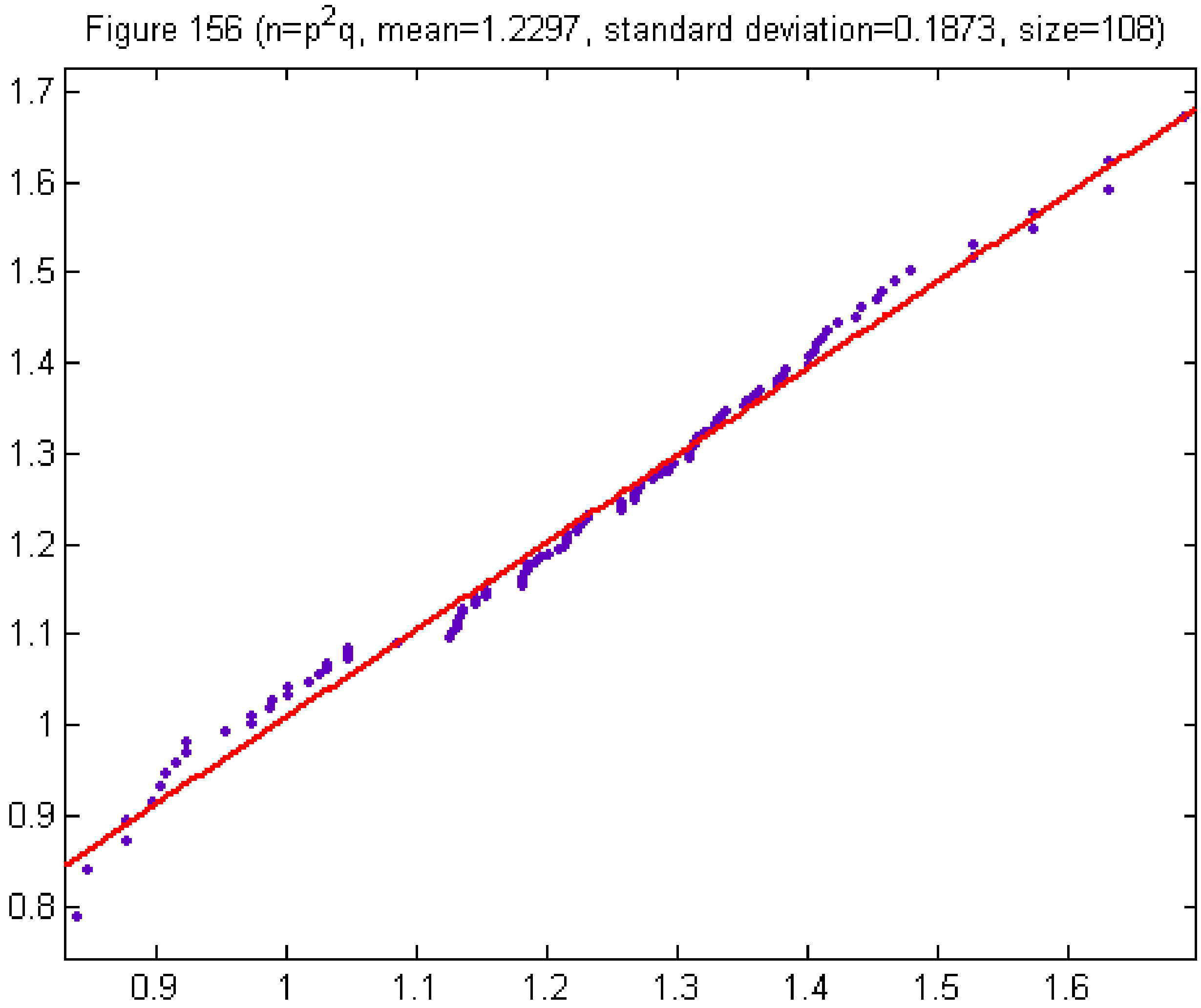


Figure 156 (n=p2q, mean=1.2297, standard deviation=0.1873, size=108)
1.7
1.6
1.5
1.4
1.3
1.2
1.1
1
0.9
0.8
0.9
1
1.1
1.2
1.3
1.4
1.5
1.6

Figure 157 ($n=p^3q$, mean=1.2765, standard deviation=0.1881, size=44)

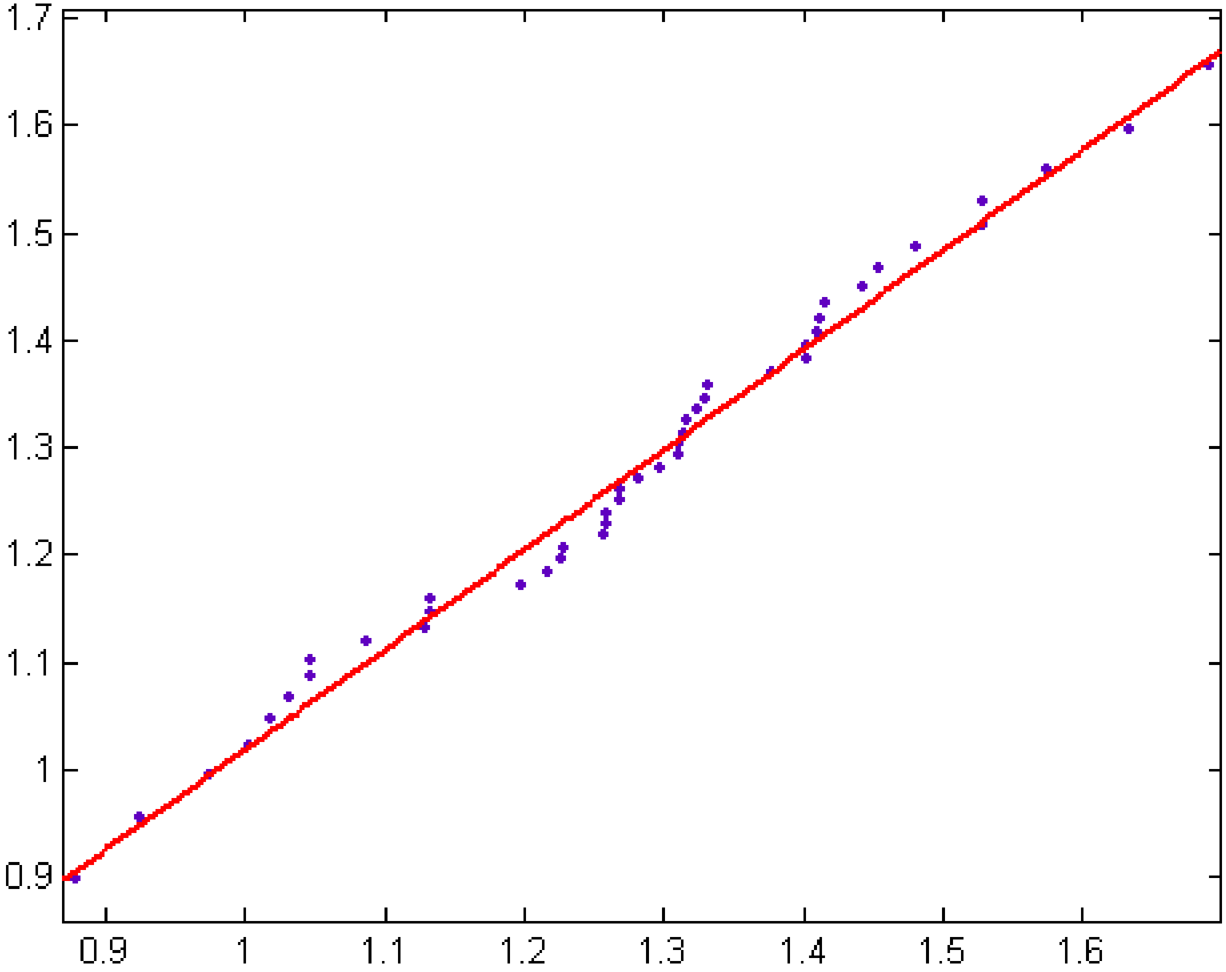
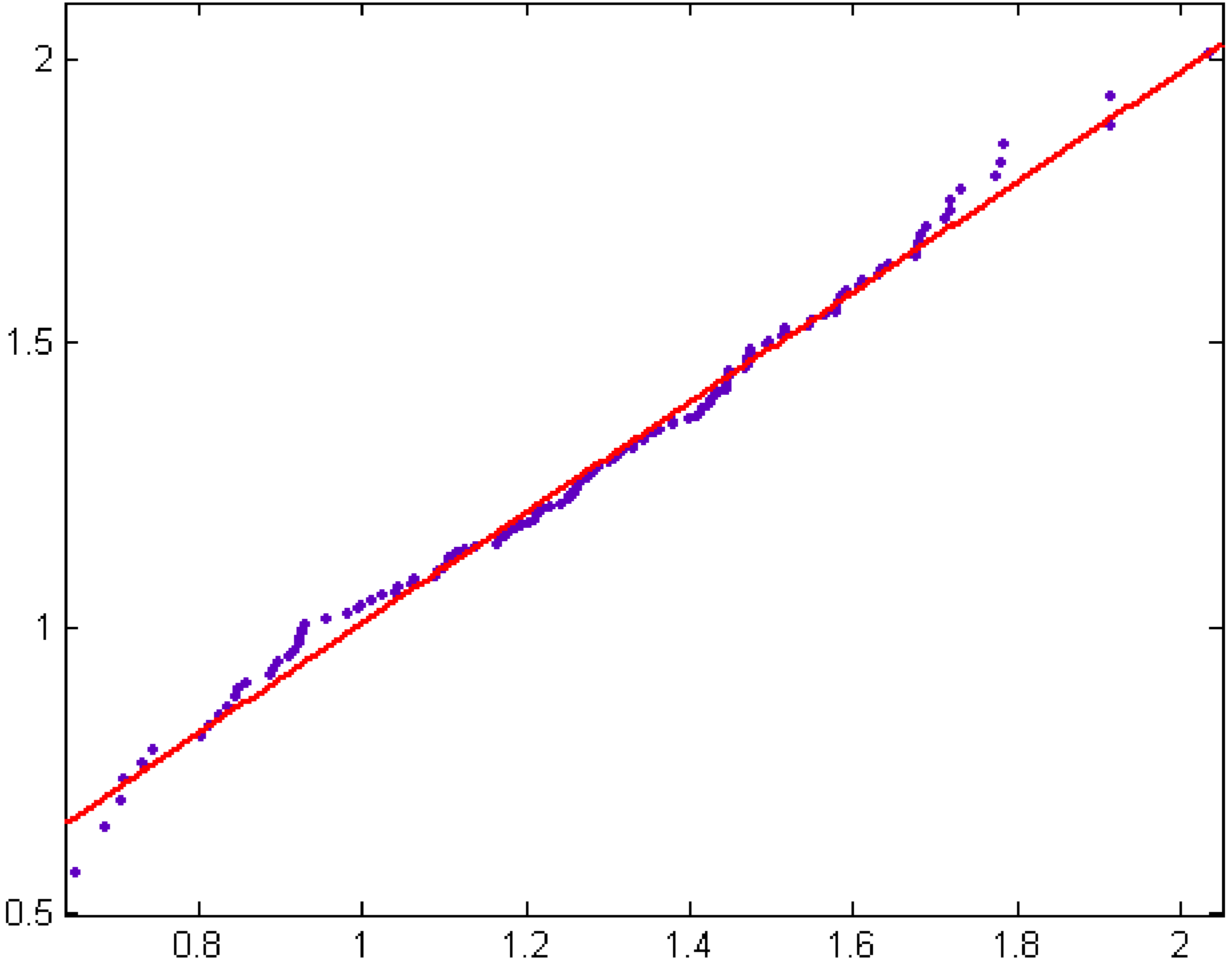


Figure 158 (n=pqr, mean=1.2890, standard deviation=0.2950, size=135)



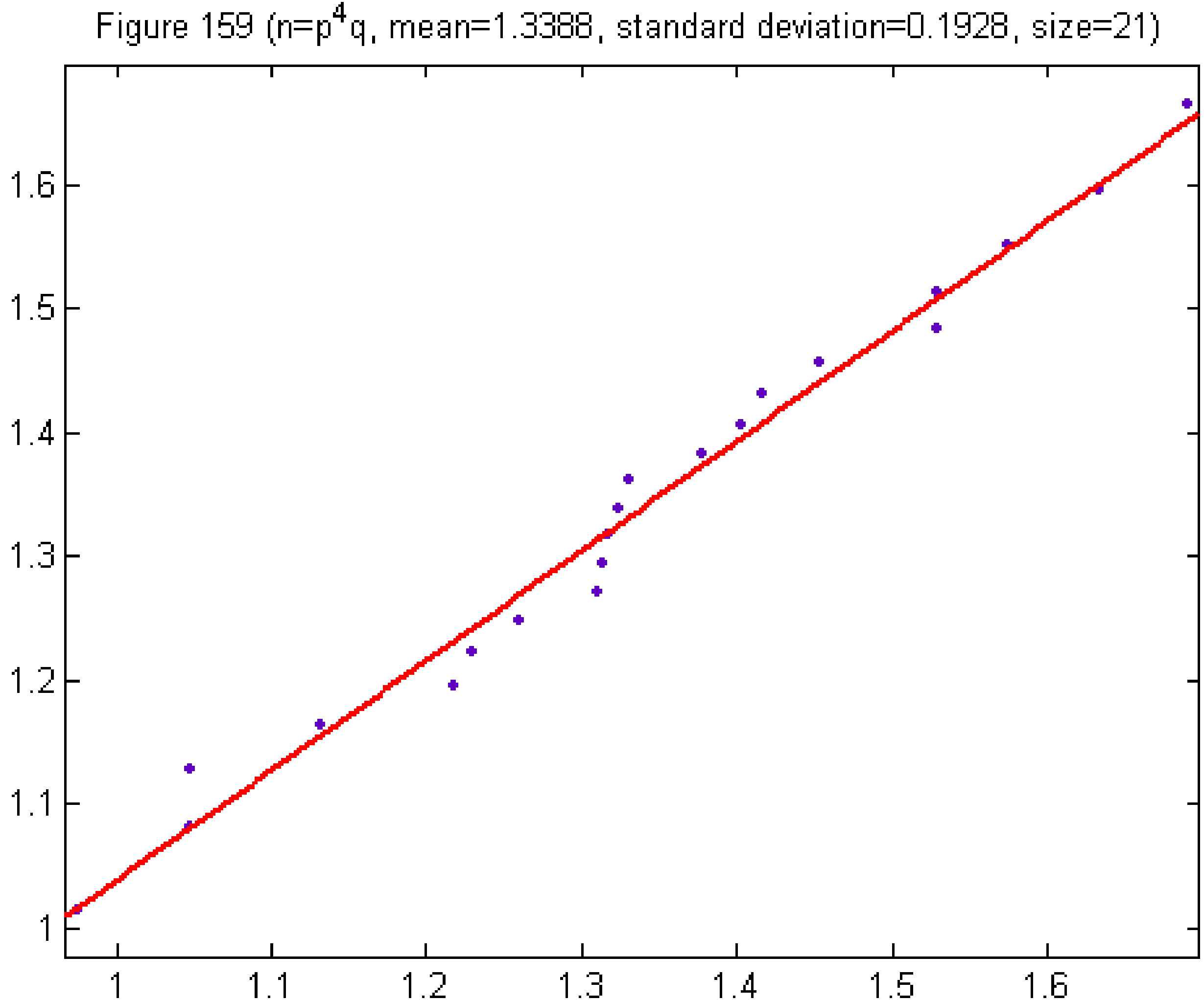


Figure 159 (n=p4q, mean=1.3388, standard deviation=0.1928, size=21)
1.6
1.5
1.4
1.3
1.2
1.1
1
1
1.1
1.2
1.3
1.4
1.5
1.6

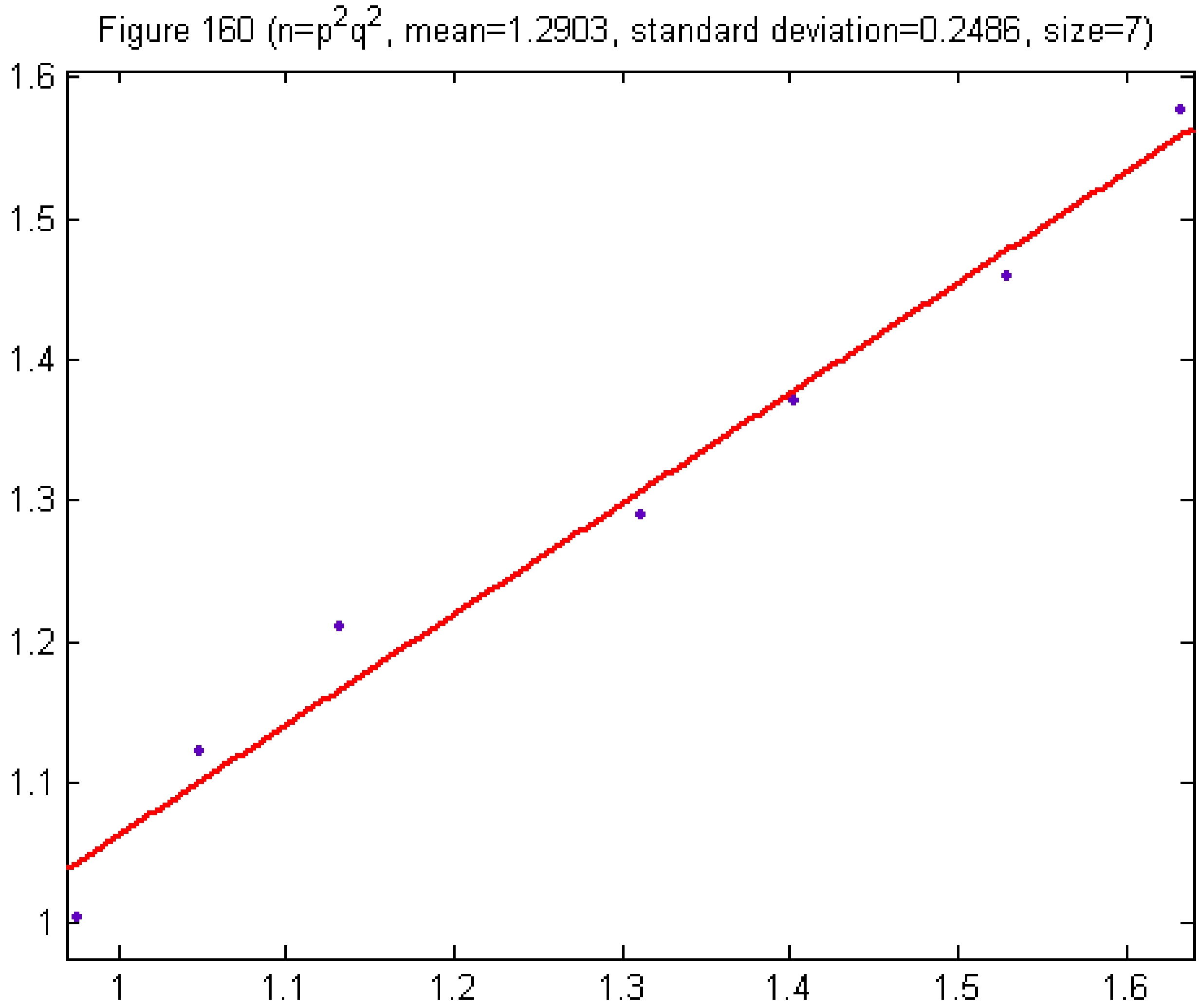


Figure 160 (n=p2q2, mean=1.2903, standard deviation=0.2486, size=7)
1.6
1.5
1.4
1.3
1.2
1.1
1
1
1.1
1.2
1.3
1.4
1.5
1.6

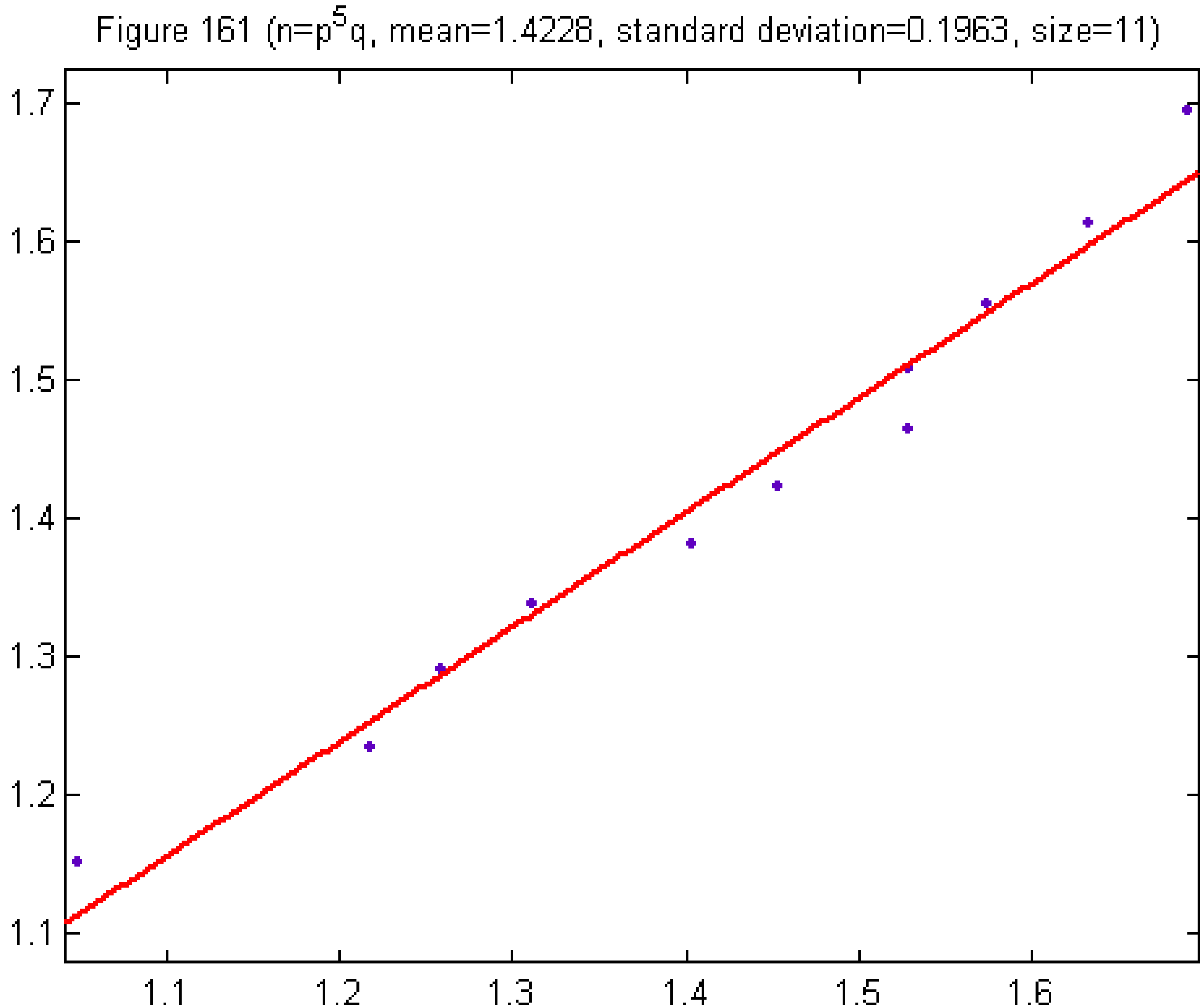
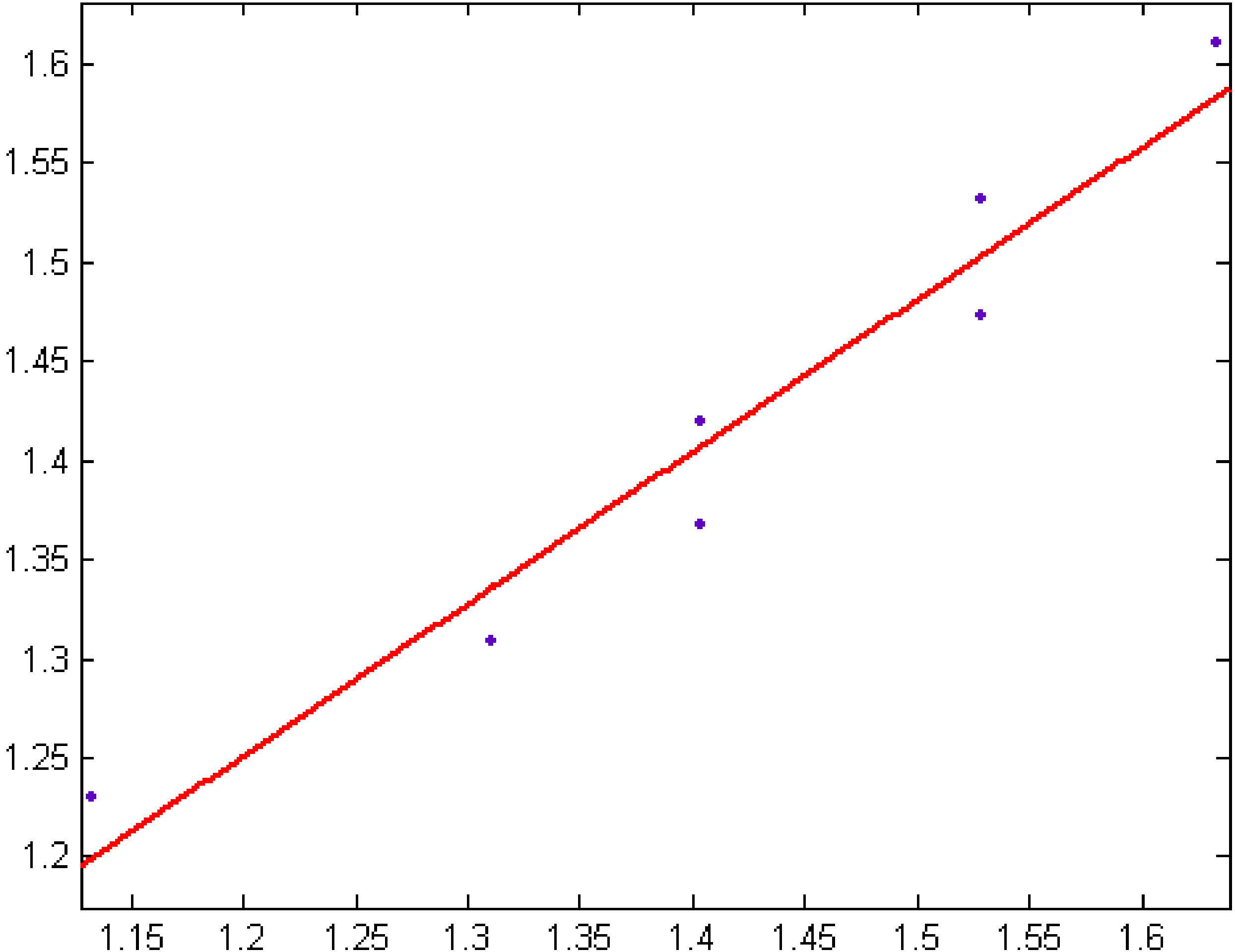


Figure 161 (n=p5q, mean=1.4228, standard deviation=0.1963, size=11)
1.7
1.6
1.5
1.4
1.3
1.2
1.1
1.1
1.2
1.3
1.4
1.5
1.6

Figure 162 ($n=p^3q^2$, mean=1.4202, standard deviation=0.1655, size=7)



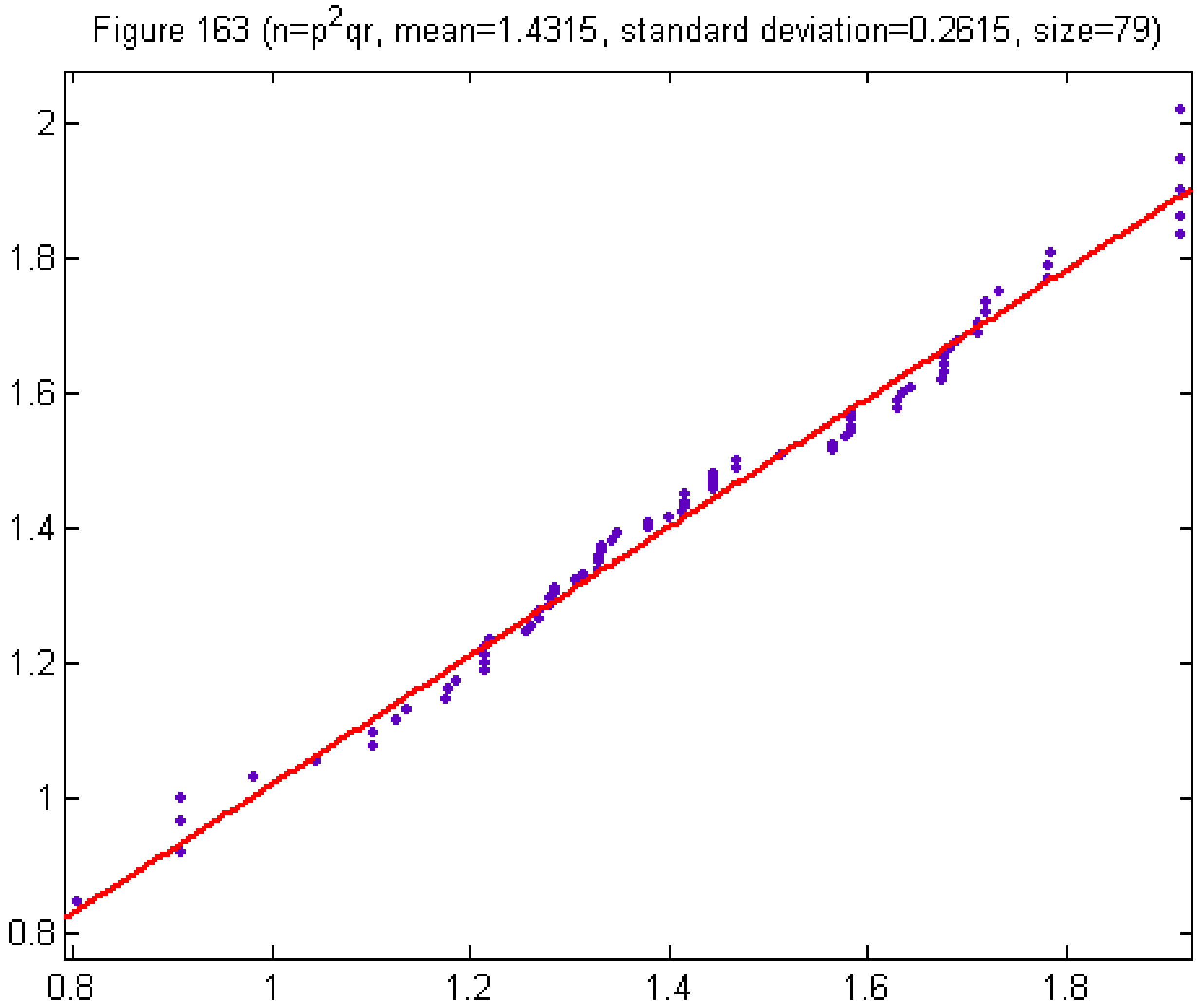


Figure 163 (n=p2qr, mean=1.4315, standard deviation=0.2615, size=79)
0.8
1
1.2
1.4
1.6
1.8
2

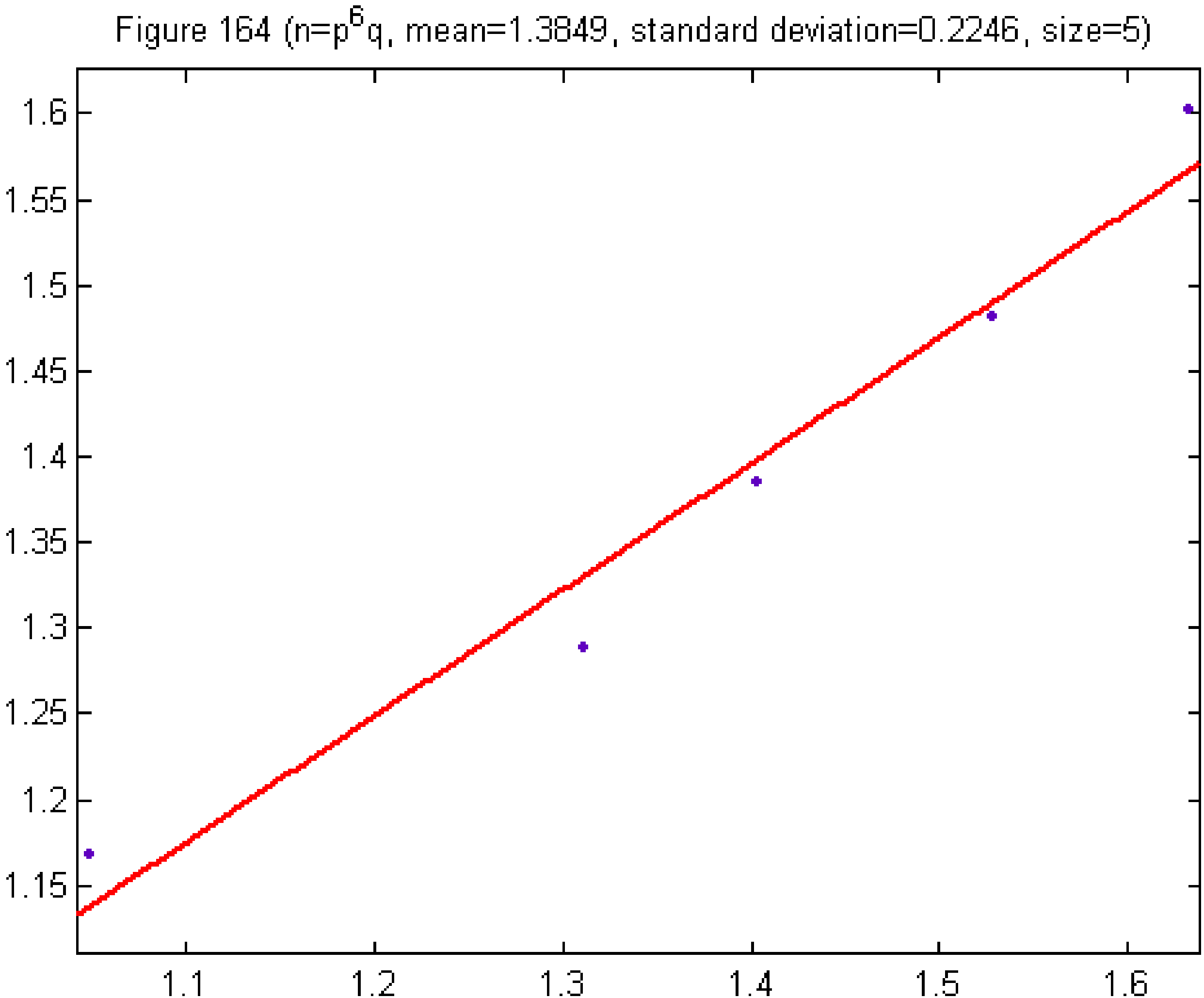


Figure 164 (n=p6q, mean=1.3849, standard deviation=0.2246, size=5)
1.6
1.55
1.5
1.45
1.4
1.35
1.3
1.25
1.2
1.15
1.1
1.2
1.3
1.4
1.5
1.6

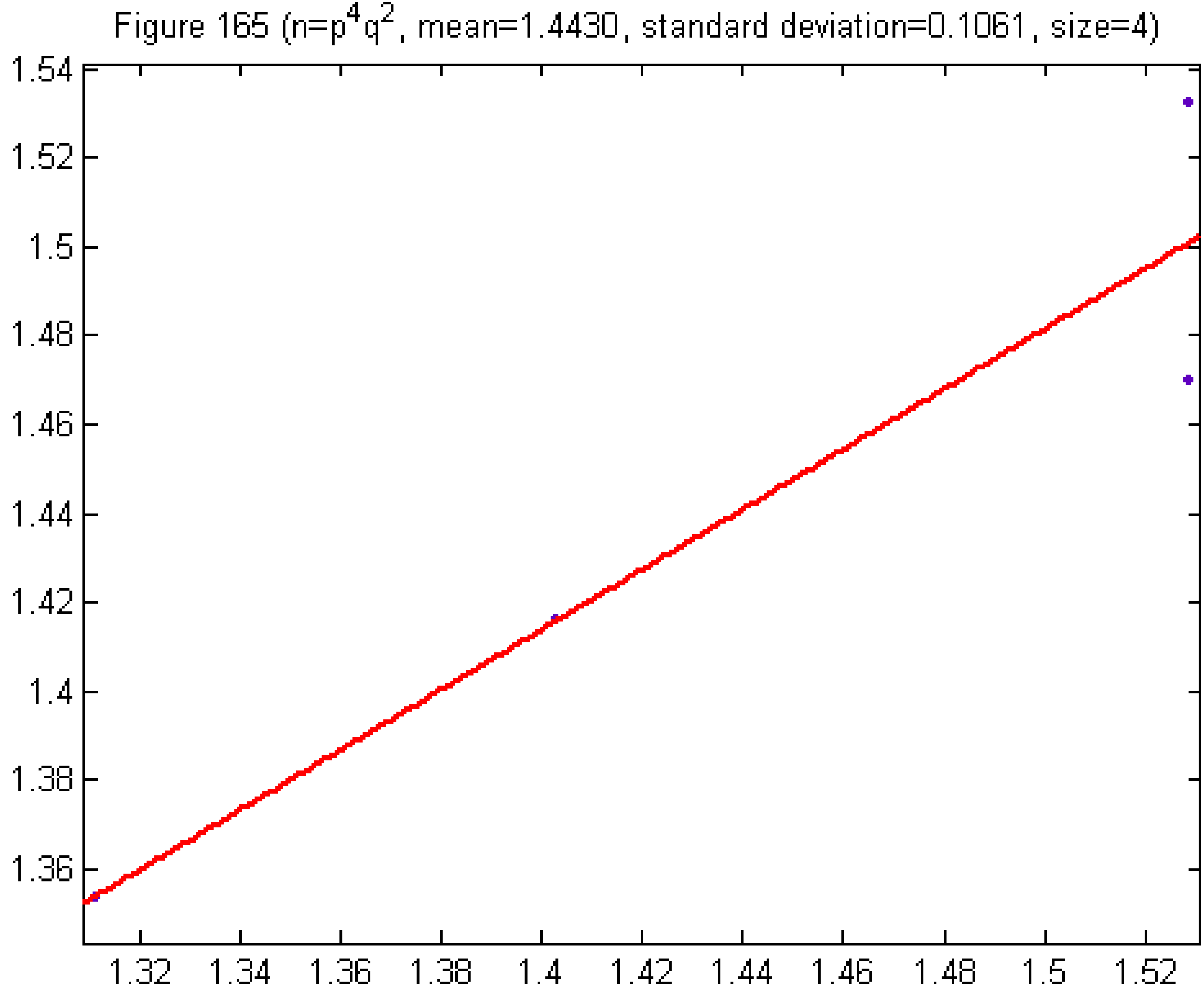


Figure 165 (n=p^4q^2, mean=1.4430, standard deviation=0.1061, size=4)
1.54
1.52
1.5
1.48
1.46
1.44
1.42
1.4
1.38
1.36
1.32
1.34
1.36
1.38
1.4
1.42
1.44
1.46
1.48
1.5
1.52

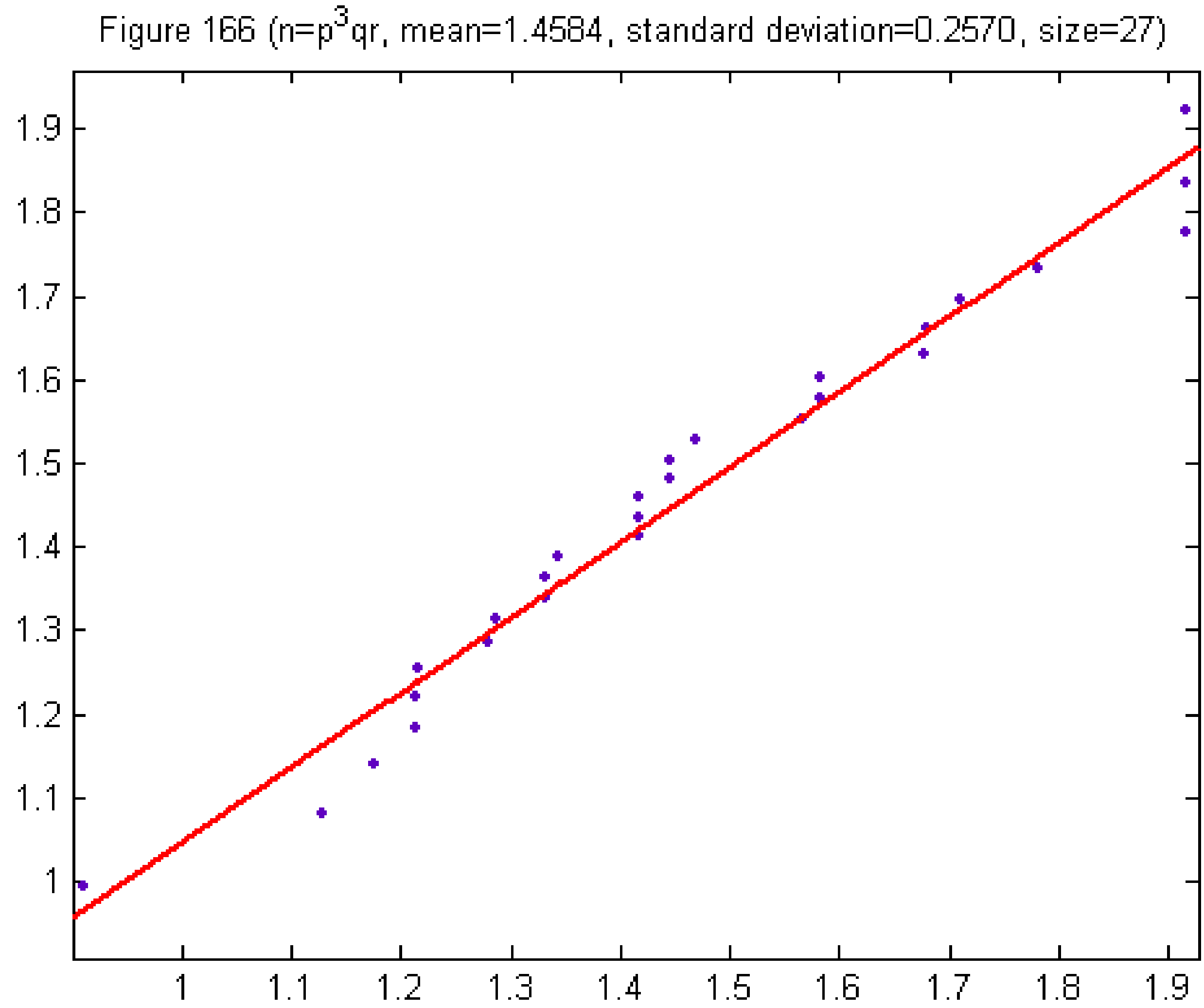


Figure 166 (n=p3qr, mean=1.4584, standard deviation=0.2570, size=27)
1.9
1.8
1.7
1.6
1.5
1.4
1.3
1.2
1.1
1
1
1.1
1.2
1.3
1.4
1.5
1.6
1.7
1.8
1.9

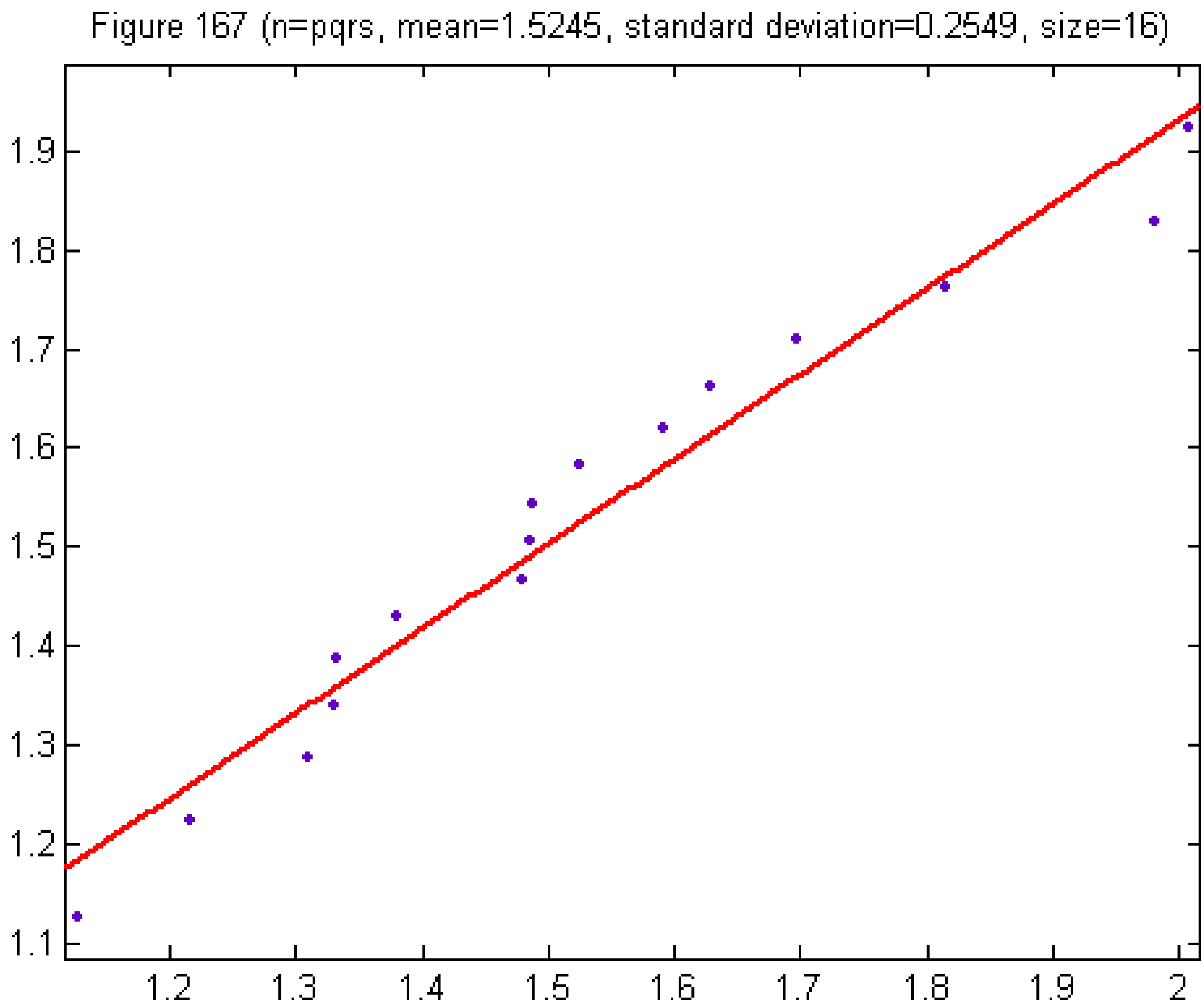


Figure 167 (n=pqrs, mean=1.5245, standard deviation=0.2549, size=16)
1.1
1.2
1.3
1.4
1.5
1.6
1.7
1.8
1.9
1.2
1.3
1.4
1.5
1.6
1.7
1.8
1.9
2

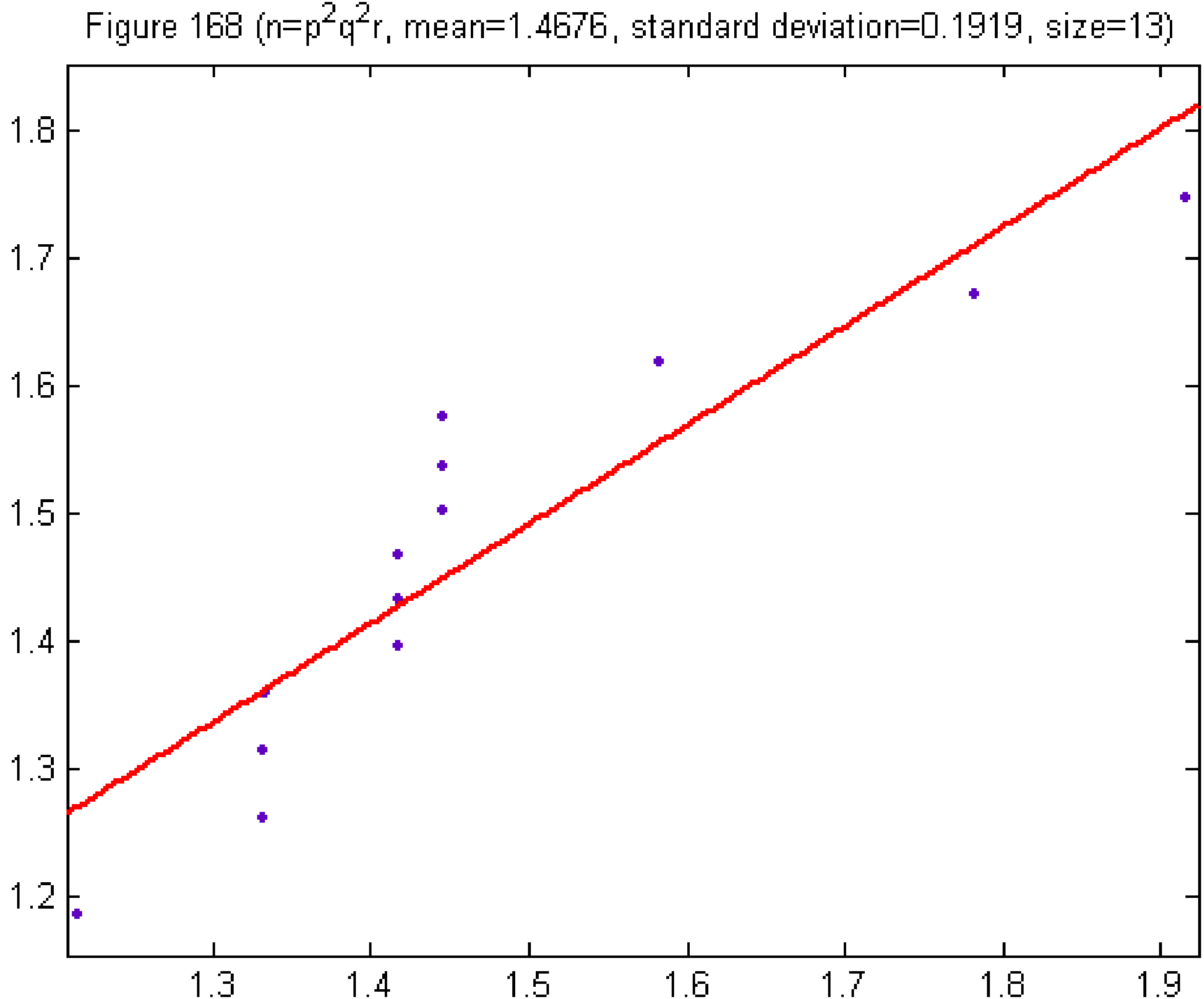


Figure 168 (n=p2q2r, mean=1.4676, standard deviation=0.1919, size=13)
1.8
1.7
1.6
1.5
1.4
1.3
1.2
1.3
1.4
1.5
1.6
1.7
1.8
1.9

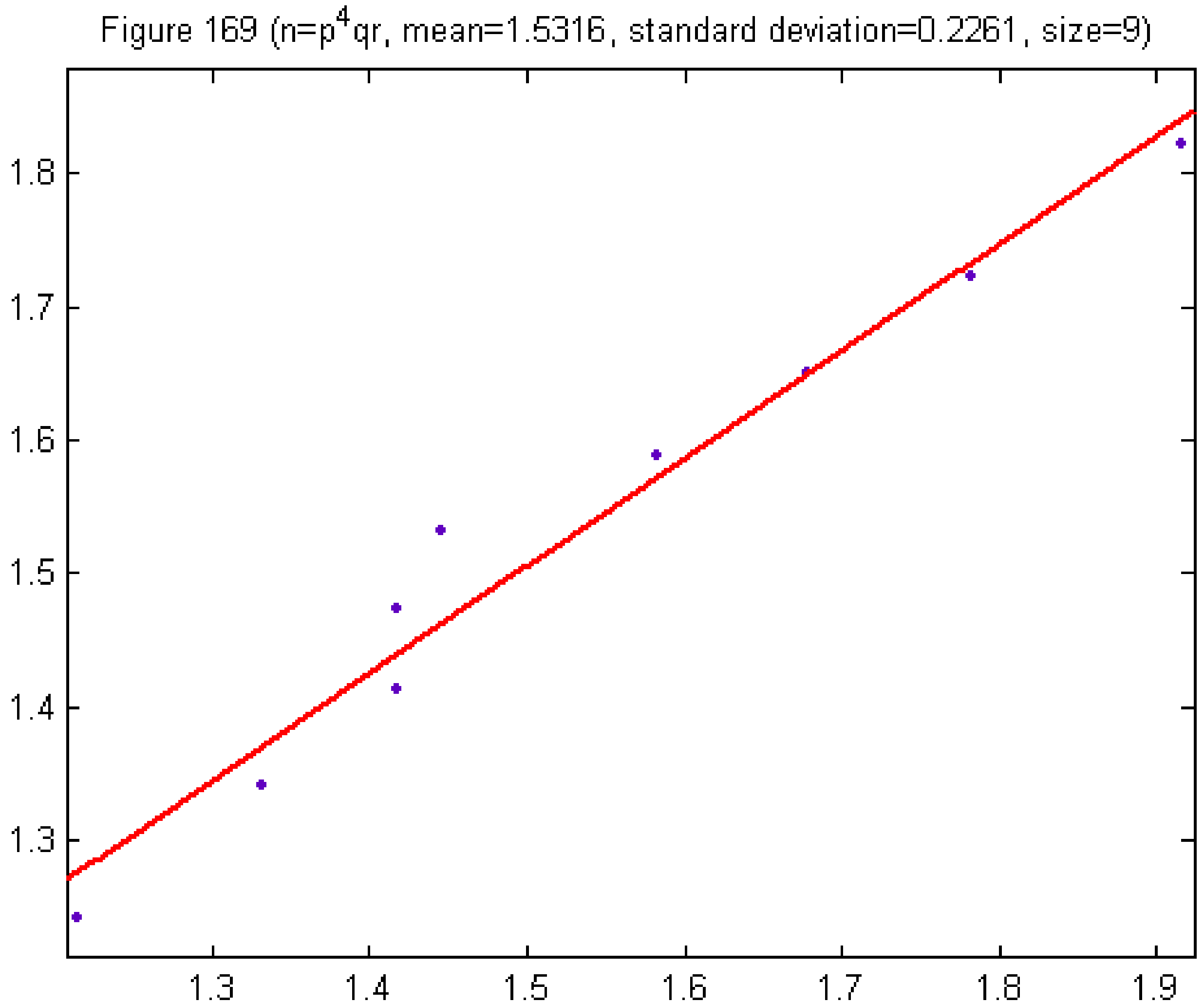


Figure 169 (n=p4qr, mean=1.5316, standard deviation=0.2261, size=9)
1.8
1.7
1.6
1.5
1.4
1.3
1.3
1.4
1.5
1.6
1.7
1.8
1.9

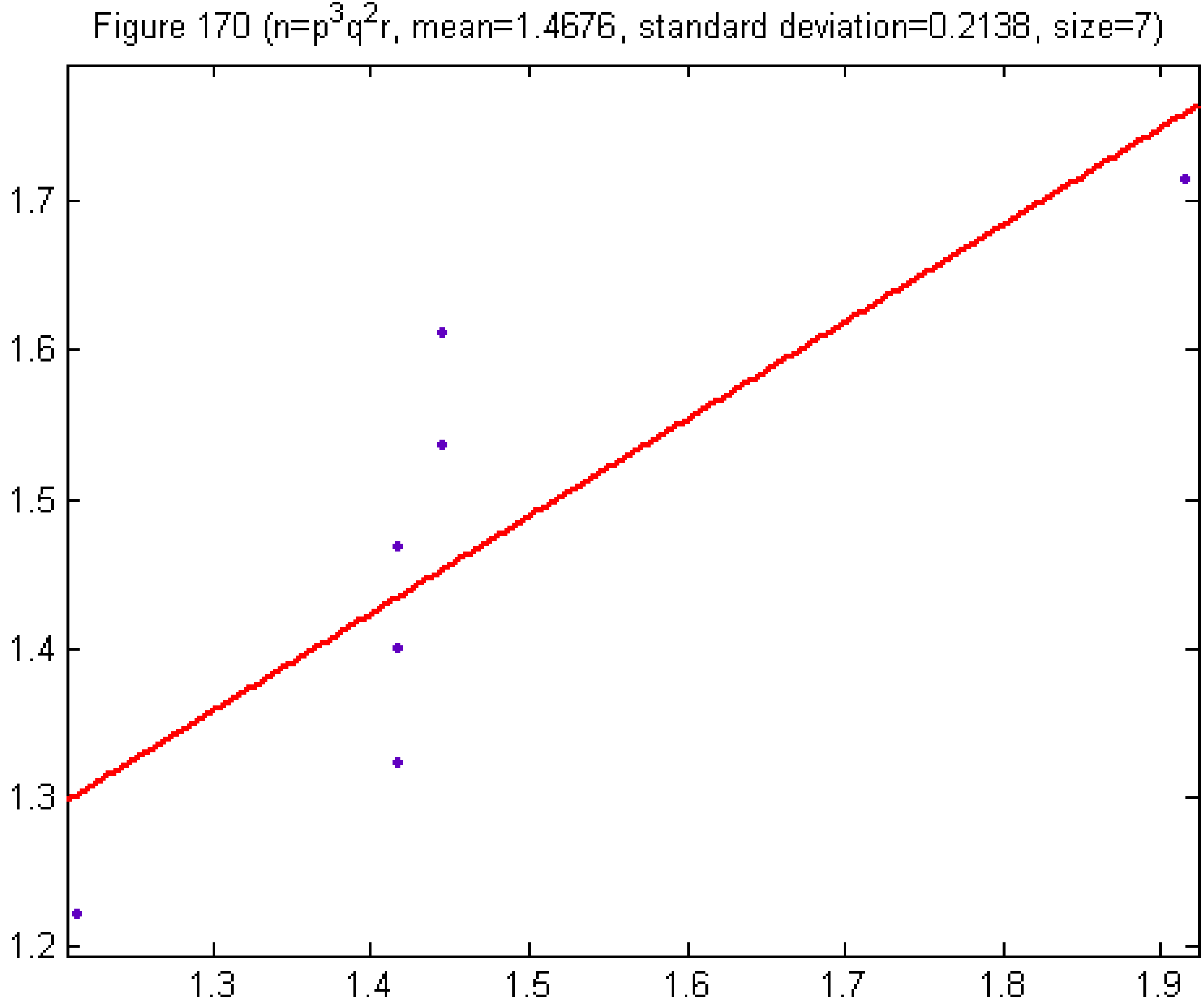


Figure 170 (n=p3q2r, mean=1.4676, standard deviation=0.2138, size=7)
1.7
1.6
1.5
1.4
1.3
1.2
1.3
1.4
1.5
1.6
1.7
1.8
1.9

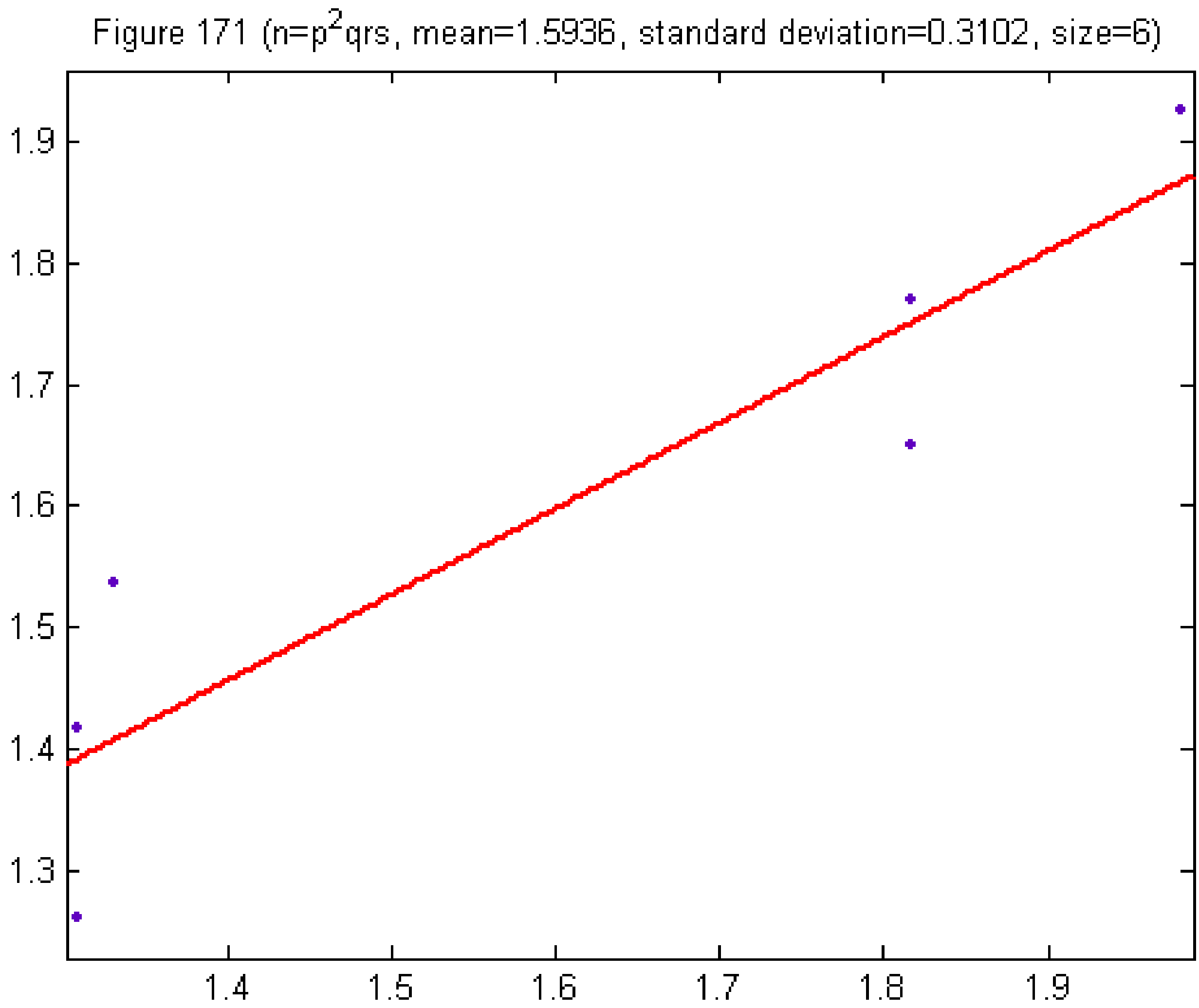


Figure 171 (n=p2qrs, mean=1.5936, standard deviation=0.3102, size=6)
1.3
1.4
1.5
1.6
1.7
1.8
1.9
1.4
1.5
1.6
1.7
1.8
1.9

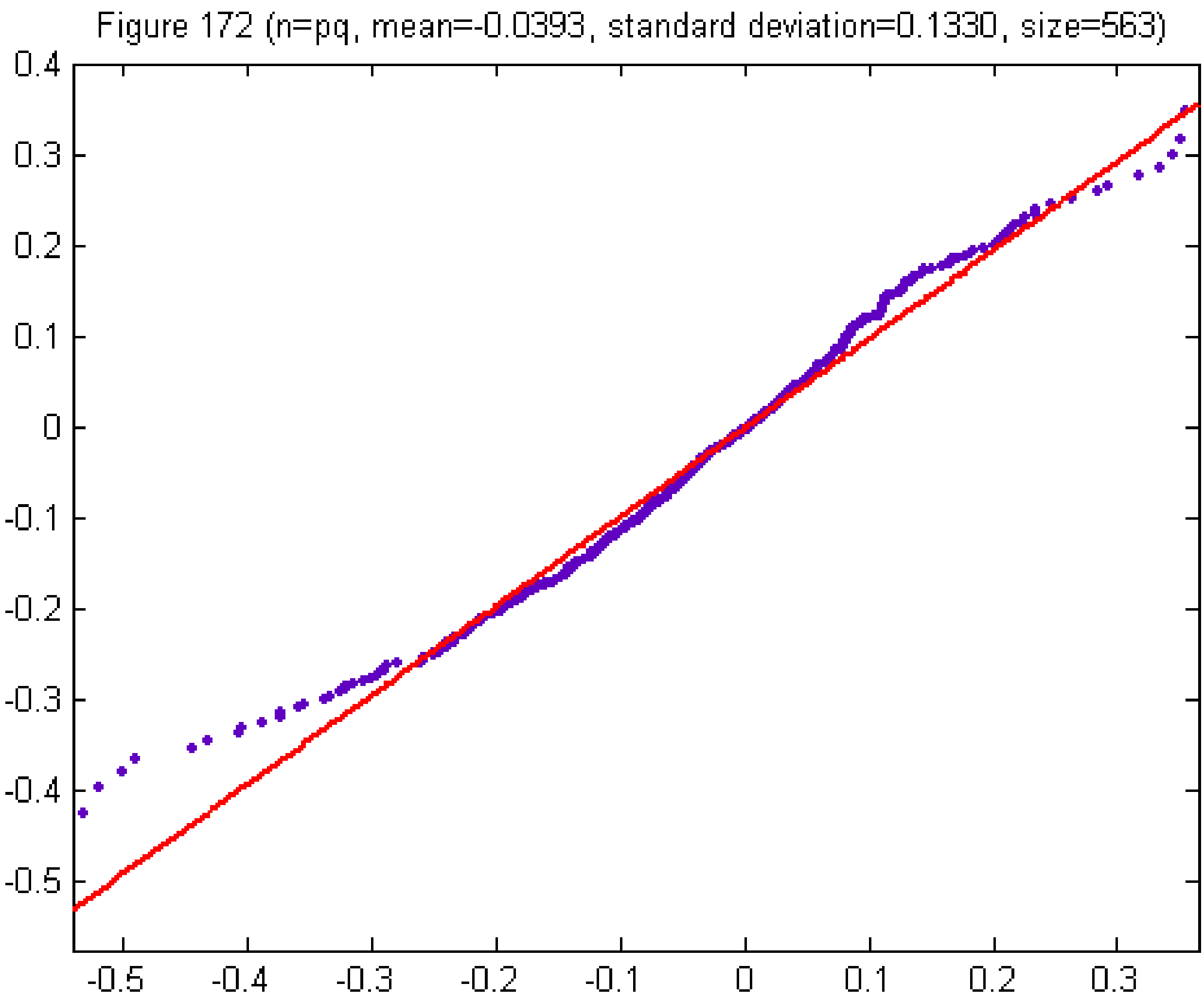


Figure 172 (n=pq, mean=-0.0393, standard deviation=0.1330, size=563)
0.4
0.3
0.2
0.1
0
-0.1
-0.2
-0.3
-0.4
-0.5
-0.5
-0.4
-0.3
-0.2
-0.1
0
0.1
0.2
0.3

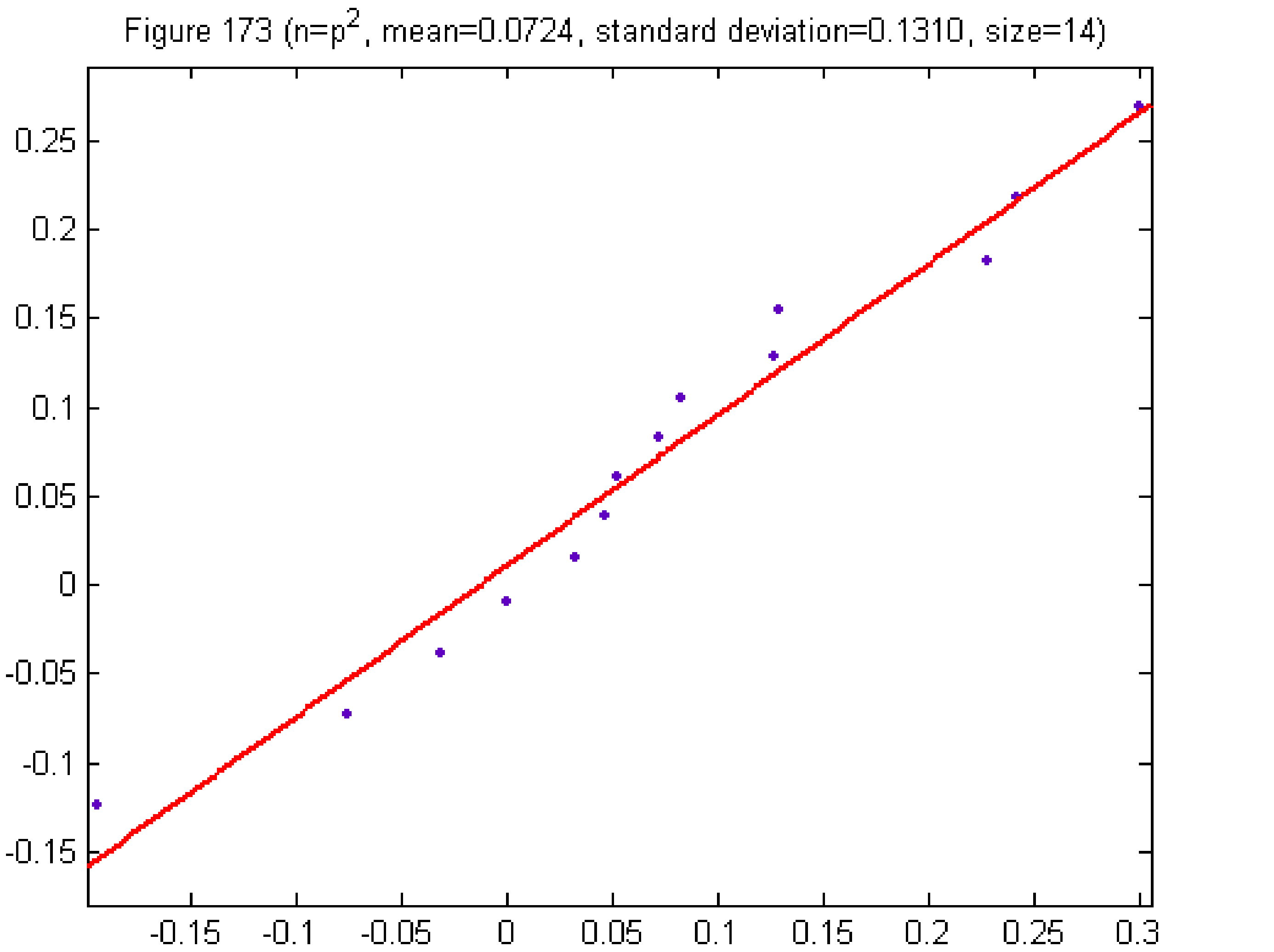


Figure 173 (n=p2, mean=0.0724, standard deviation=0.1310, size=14)
0.25
0.2
0.15
0.1
0.05
0
-0.05
-0.1
-0.15
-0.15
-0.1
-0.05
0
0.05
0.1
0.15
0.2
0.25
0.3

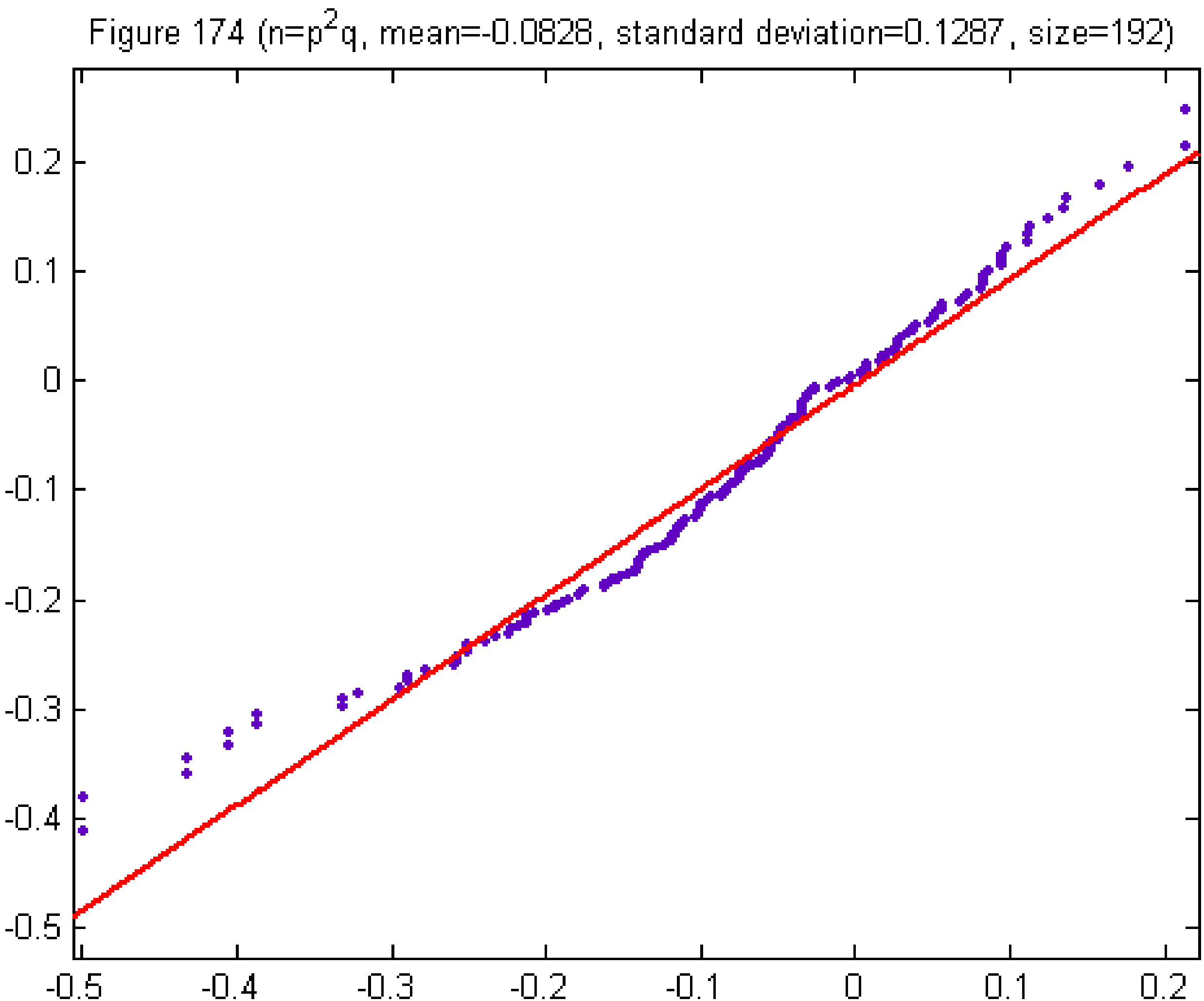
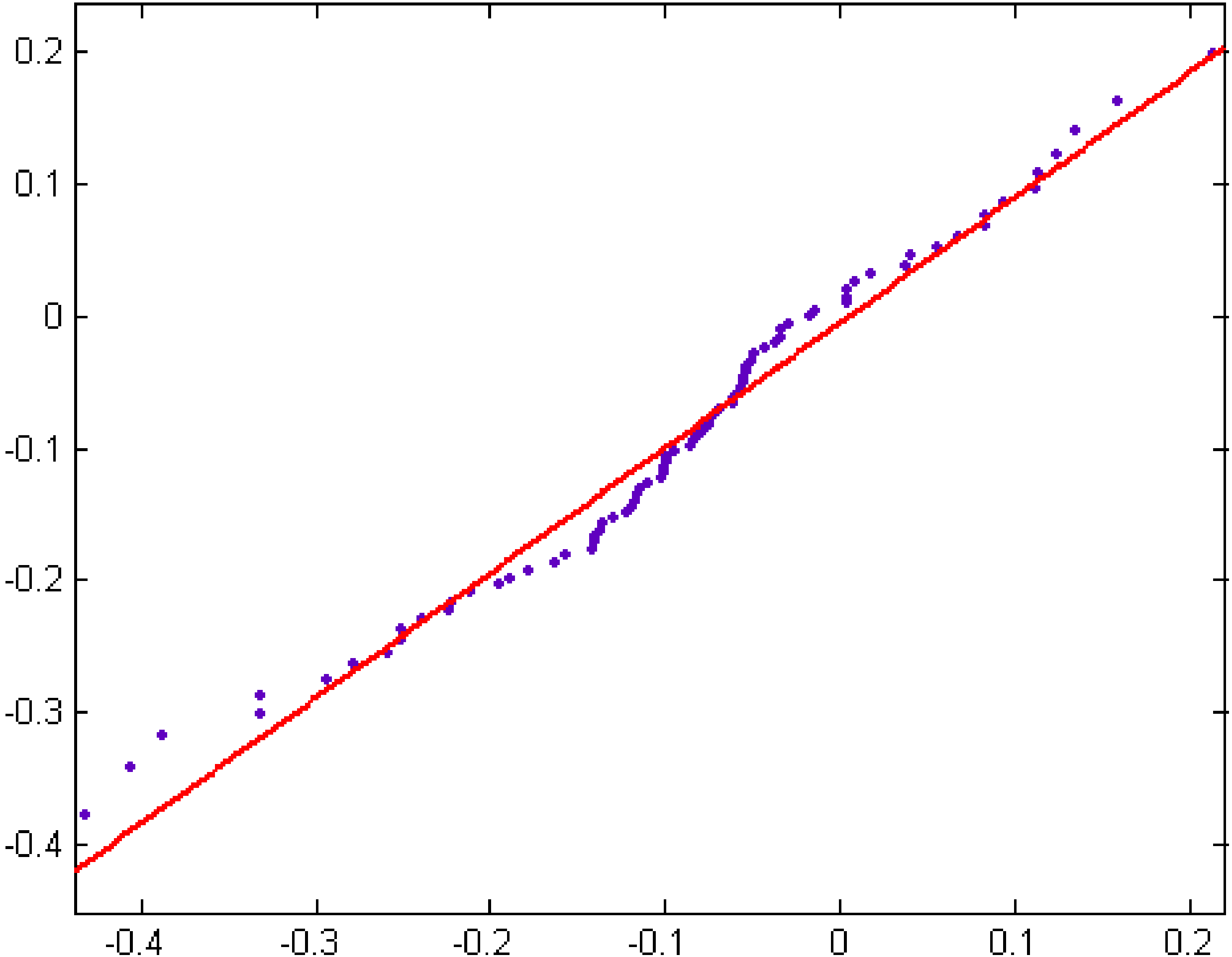


Figure 174 (n=p^2q, mean=-0.0828, standard deviation=0.1287, size=192)
0.2
0.1
0
-0.1
-0.2
-0.3
-0.4
-0.5
-0.5
-0.4
-0.3
-0.2
-0.1
0
0.1
0.2

Figure 175 ($n=p^3q$, mean=-0.0890, standard deviation=0.1283, size=80)



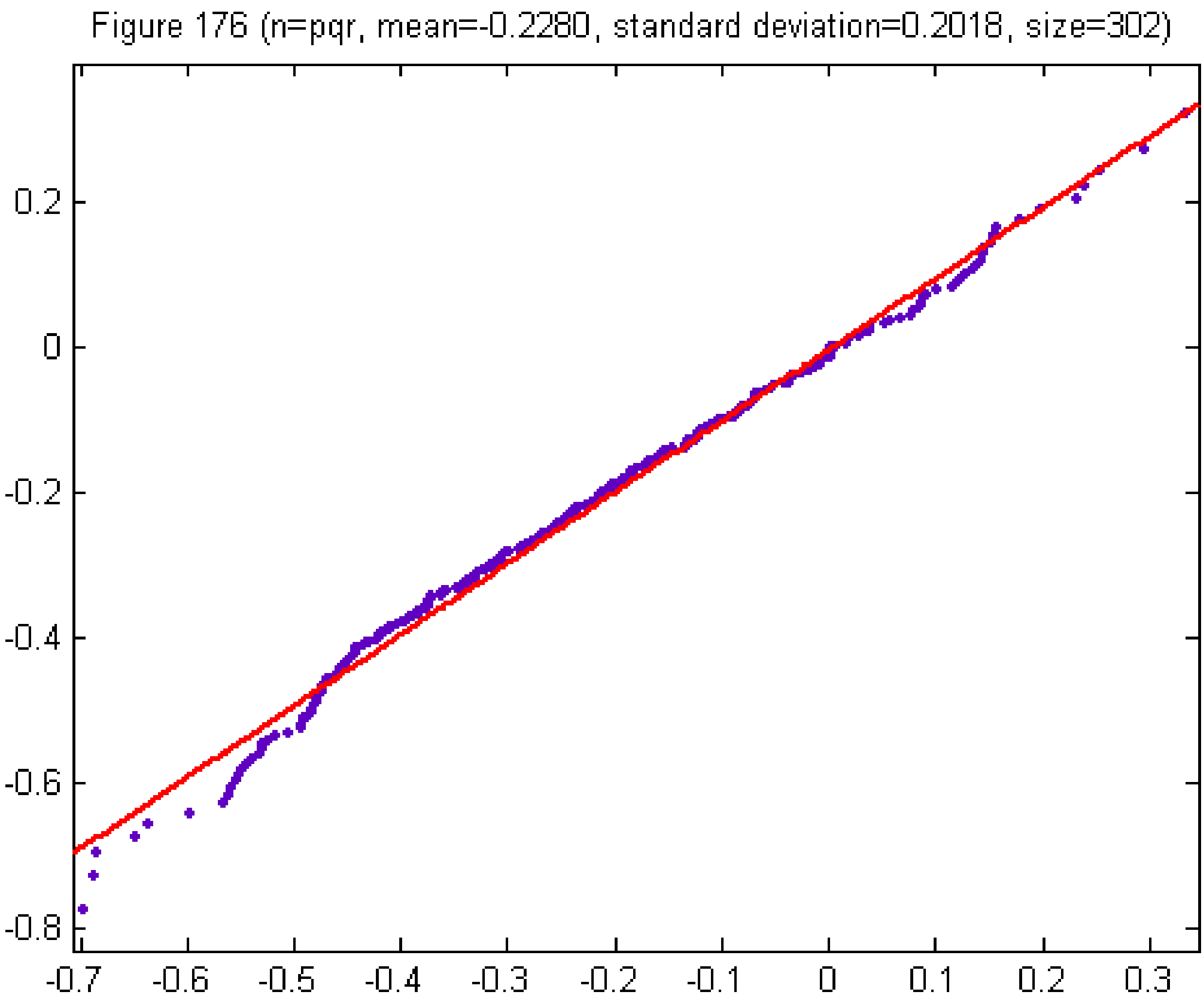


Figure 176 (n=pqr, mean=-0.2280, standard deviation=0.2018, size=302)
0.2
0
-0.2
-0.4
-0.6
-0.8
-0.7
-0.6
-0.5
-0.4
-0.3
-0.2
-0.1
0
0.1
0.2
0.3

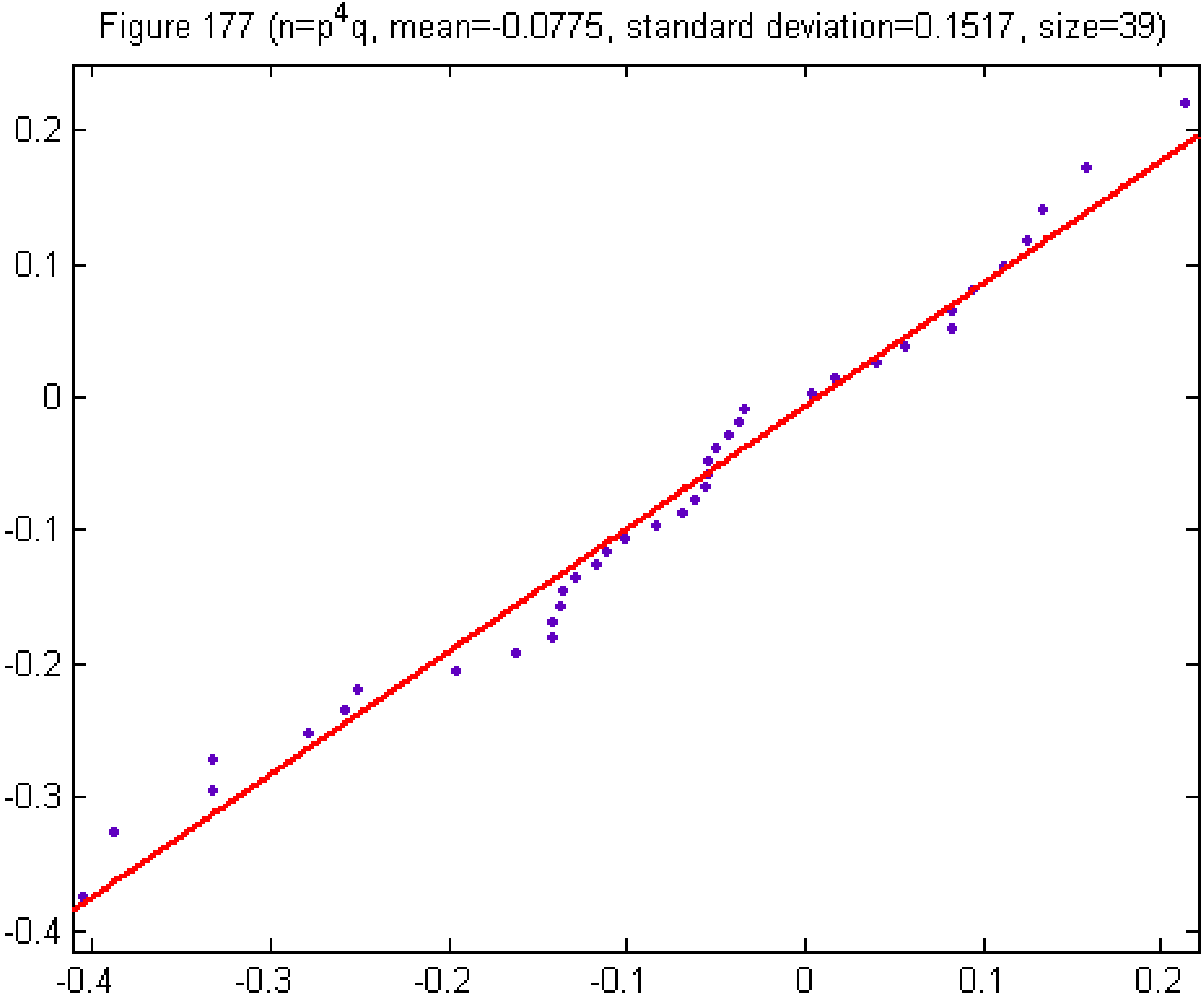
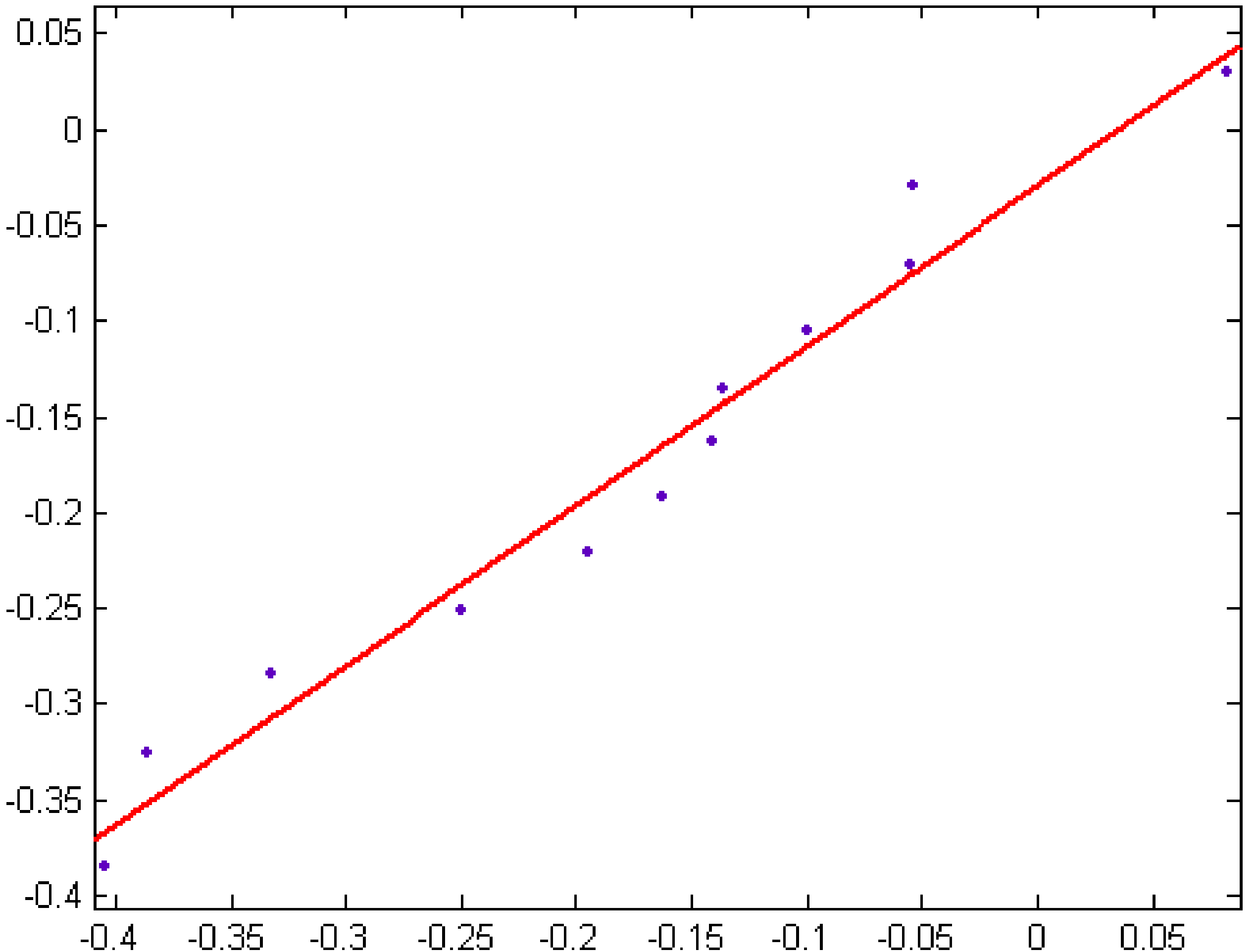
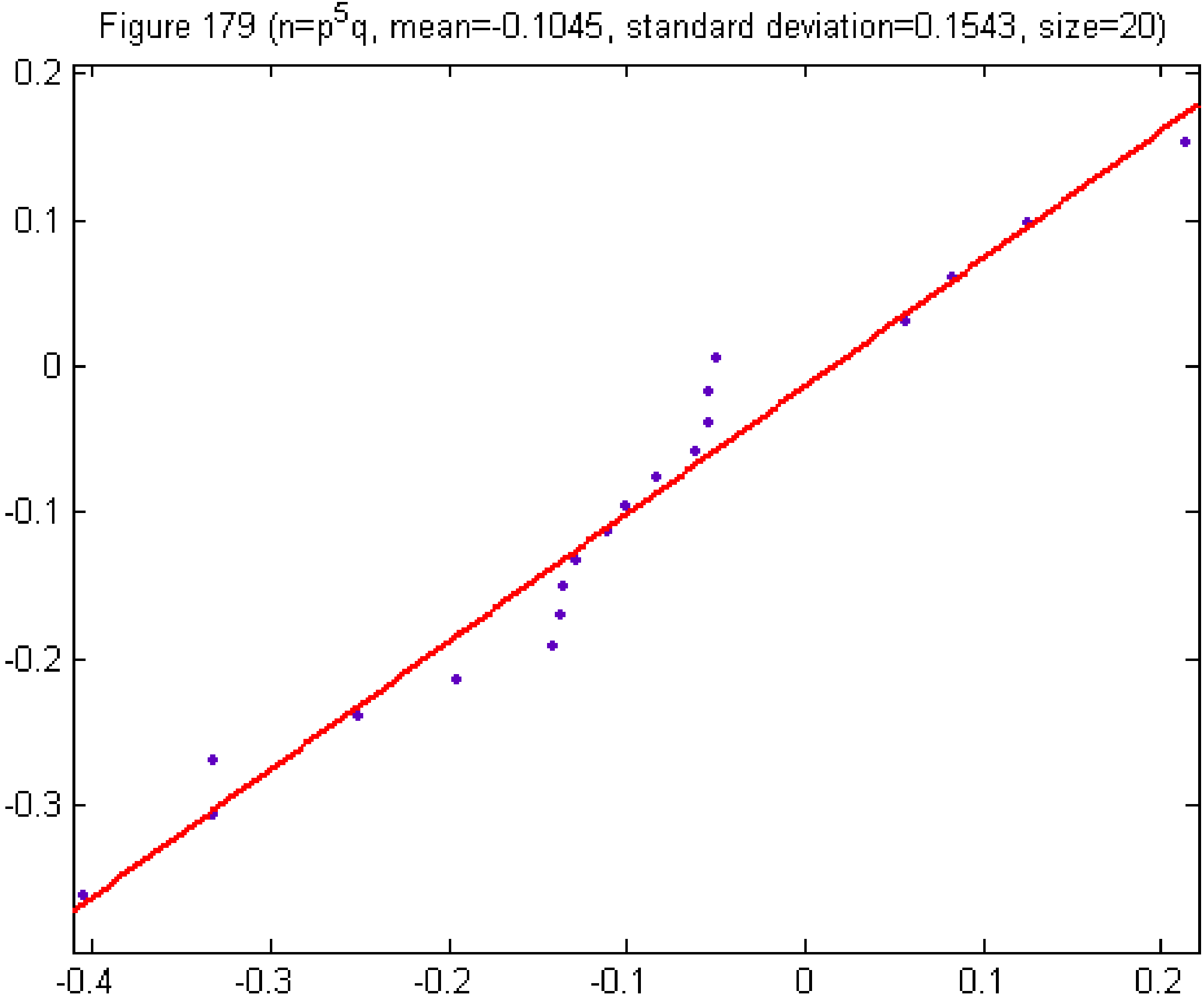
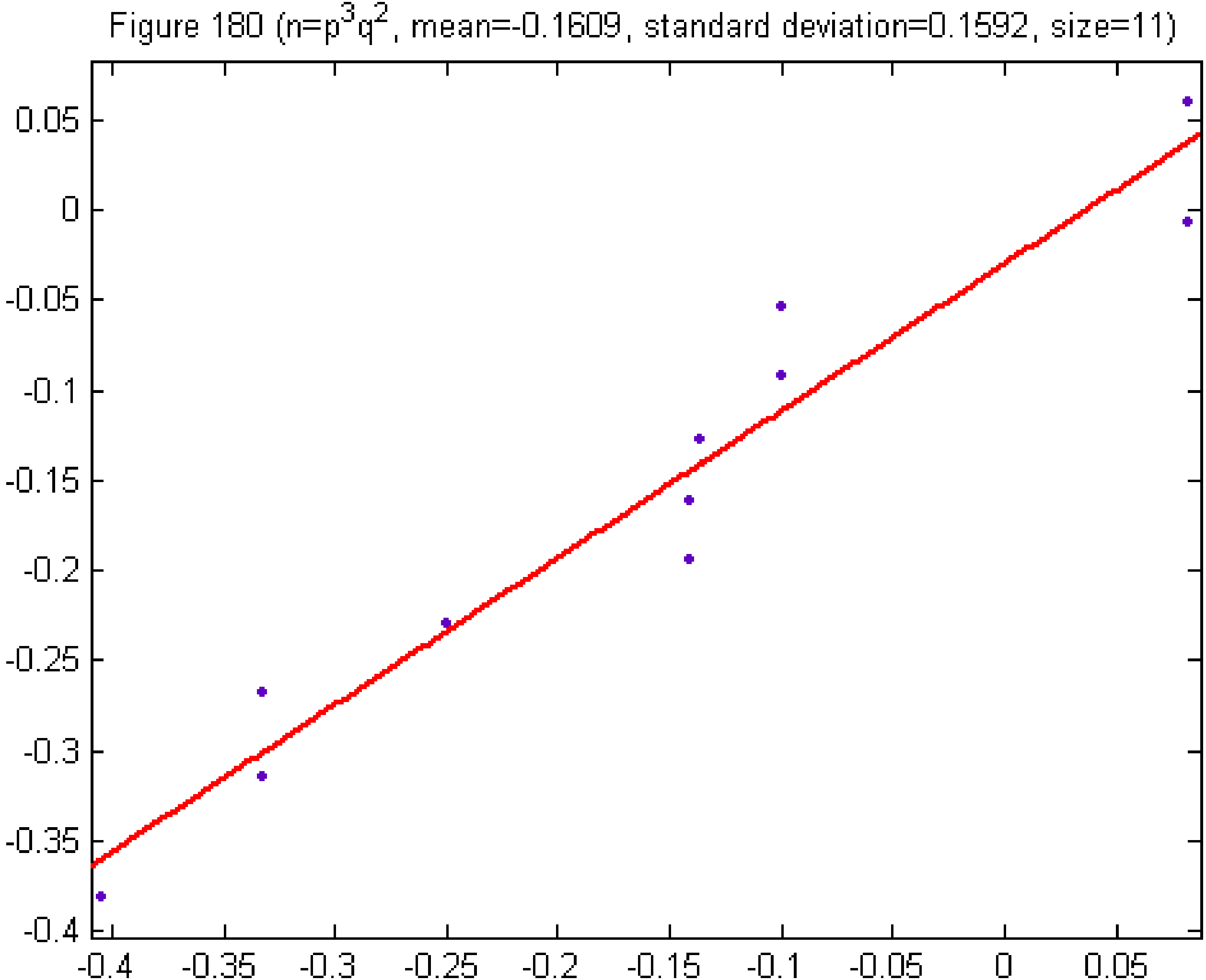


Figure 177 (n=p4q, mean=-0.0775, standard deviation=0.1517, size=39)
0.2
0.1
0
-0.1
-0.2
-0.3
-0.4
-0.4
-0.3
-0.2
-0.1
0
0.1
0.2

Figure 178 ($n=p^2q^2$, mean=-0.1776, standard deviation=0.1453, size=12)







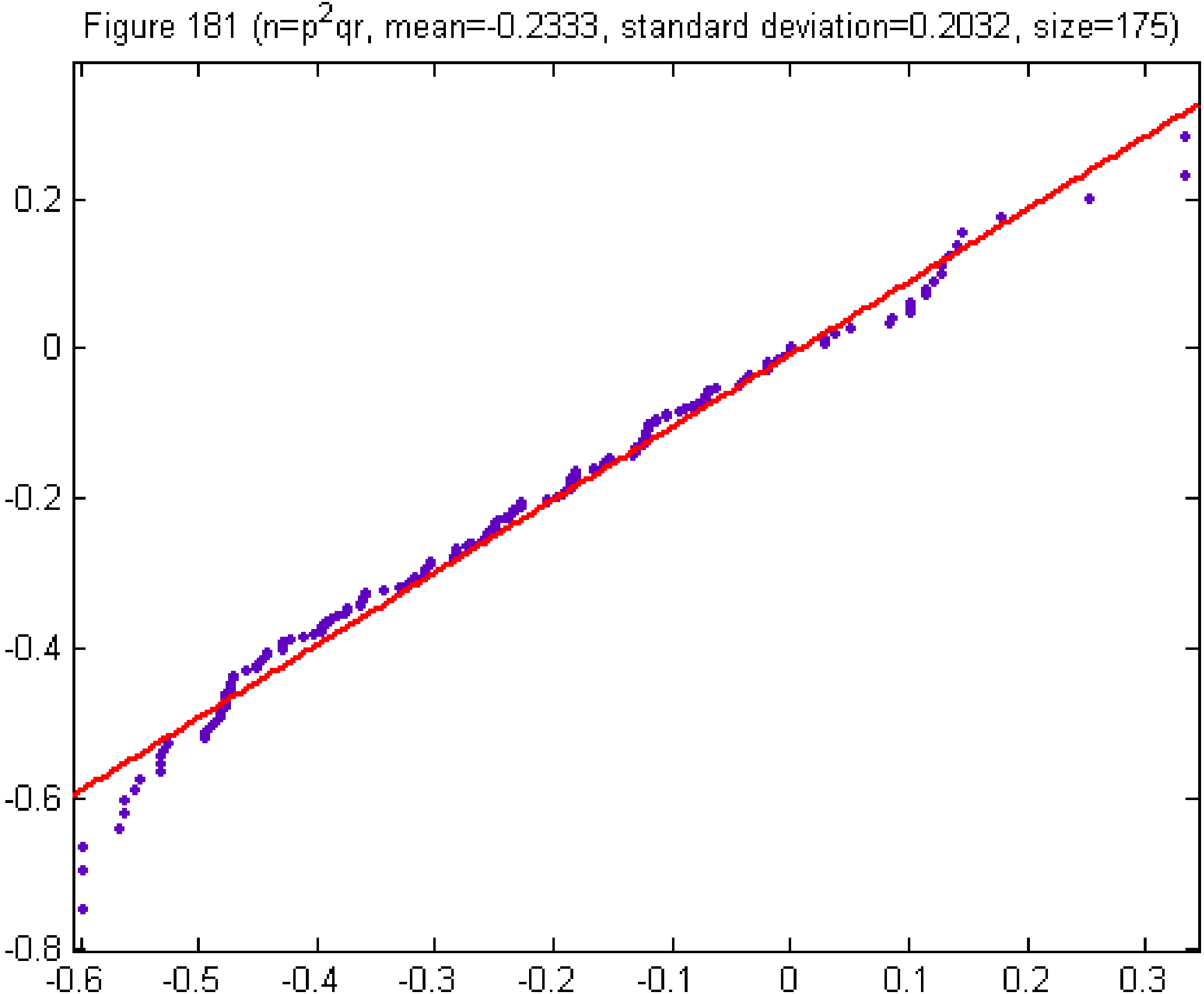


Figure 181 (n=p2qr, mean=-0.2333, standard deviation=0.2032, size=175)
0.2
0
-0.2
-0.4
-0.6
-0.8
-0.6
-0.5
-0.4
-0.3
-0.2
-0.1
0
0.1
0.2
0.3

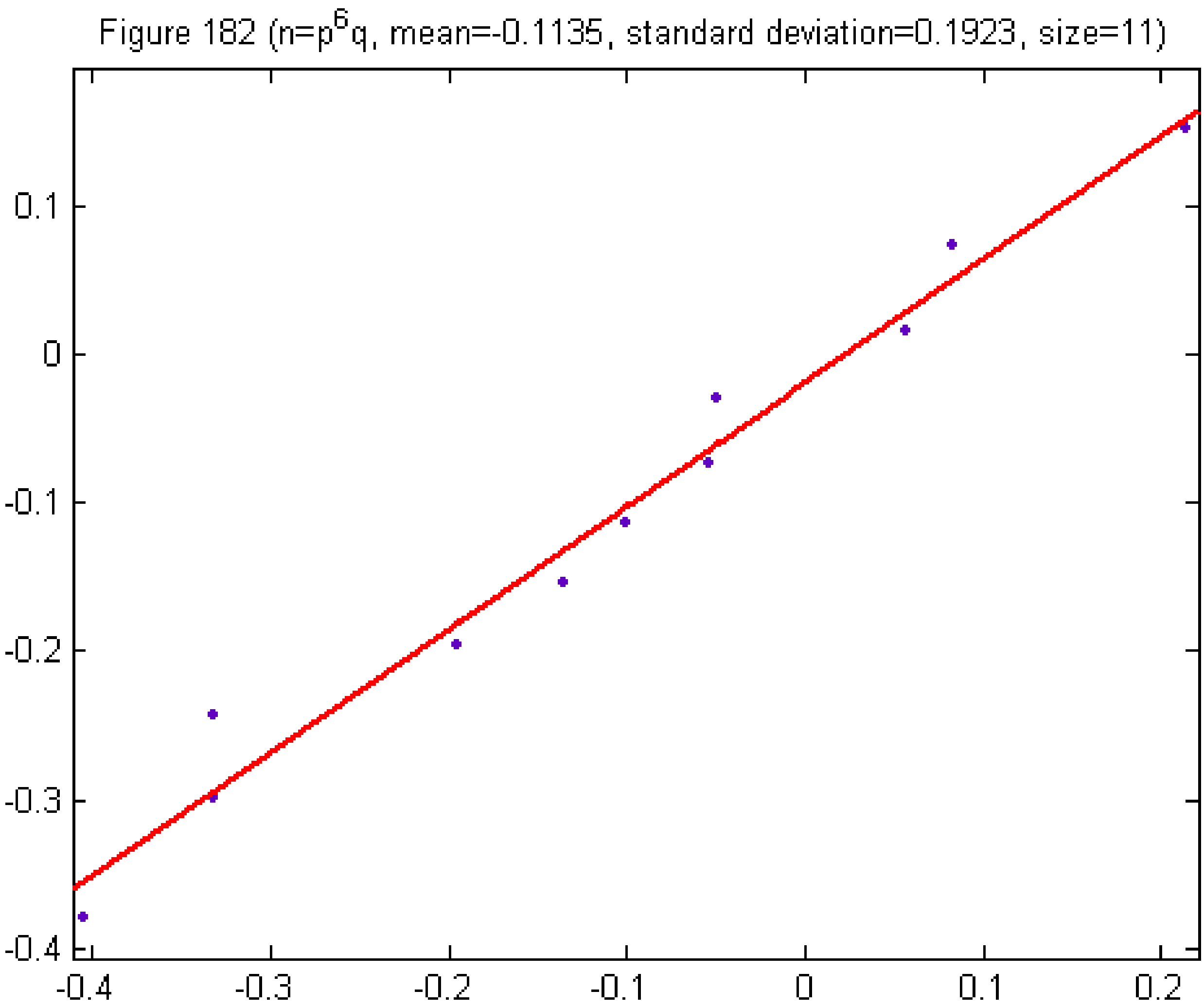


Figure 182 (n=p6q, mean=-0.1135, standard deviation=0.1923, size=11)
0.1
0
-0.1
-0.2
-0.3
-0.4
-0.4
-0.3
-0.2
-0.1
0
0.1
0.2

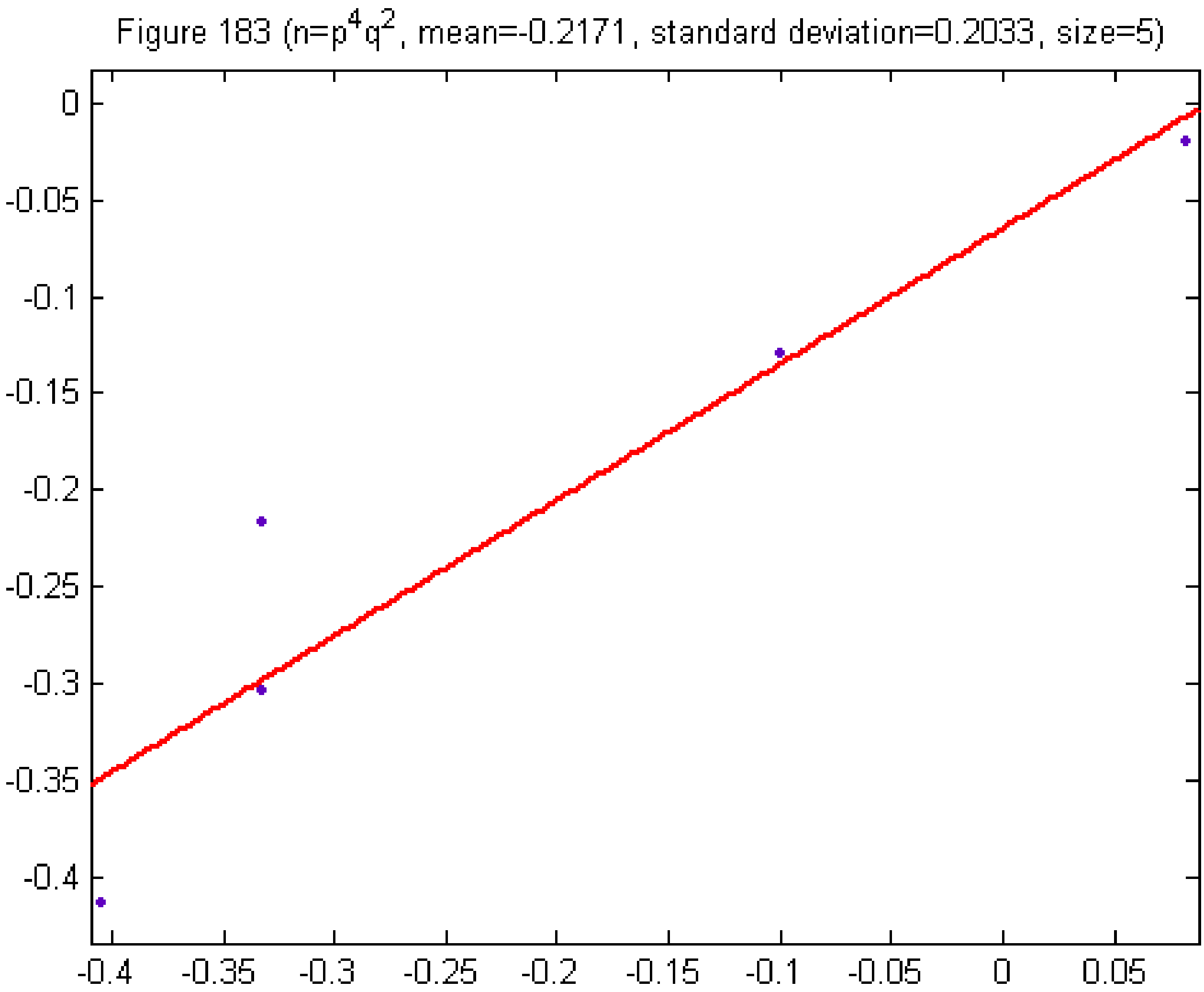


Figure 183 ($n=p^4q^2$, mean=-0.2171, standard deviation=0.2033, size=5)
0
-0.05
-0.1
-0.15
-0.2
-0.25
-0.3
-0.35
-0.4
-0.4
-0.35
-0.3
-0.25
-0.2
-0.15
-0.1
-0.05
0
0.05

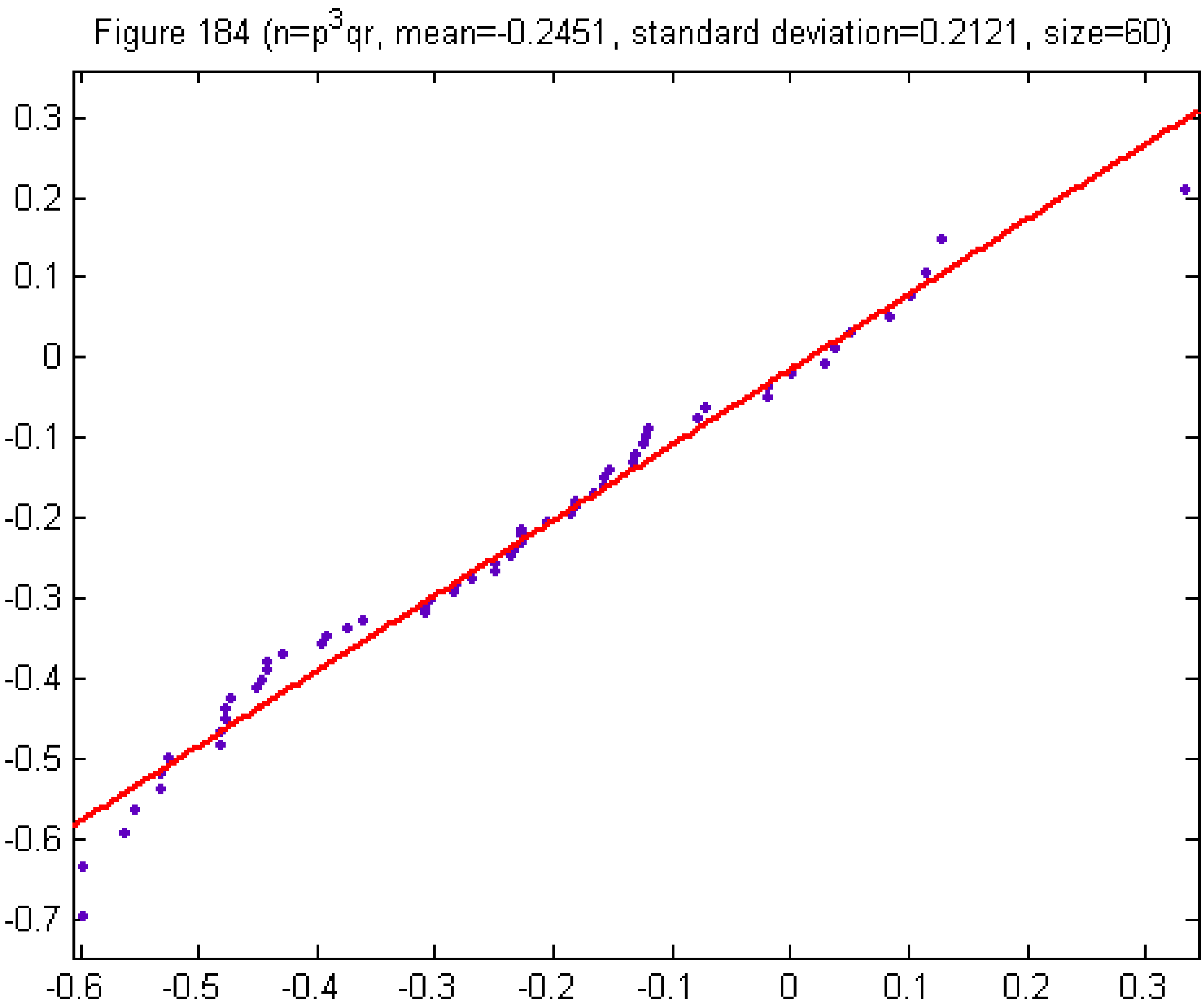


Figure 184 (n=p³qr, mean=-0.2451, standard deviation=0.2121, size=60)
0.3
0.2
0.1
0
-0.1
-0.2
-0.3
-0.4
-0.5
-0.6
-0.7
-0.6
-0.5
-0.4
-0.3
-0.2
-0.1
0
0.1
0.2
0.3

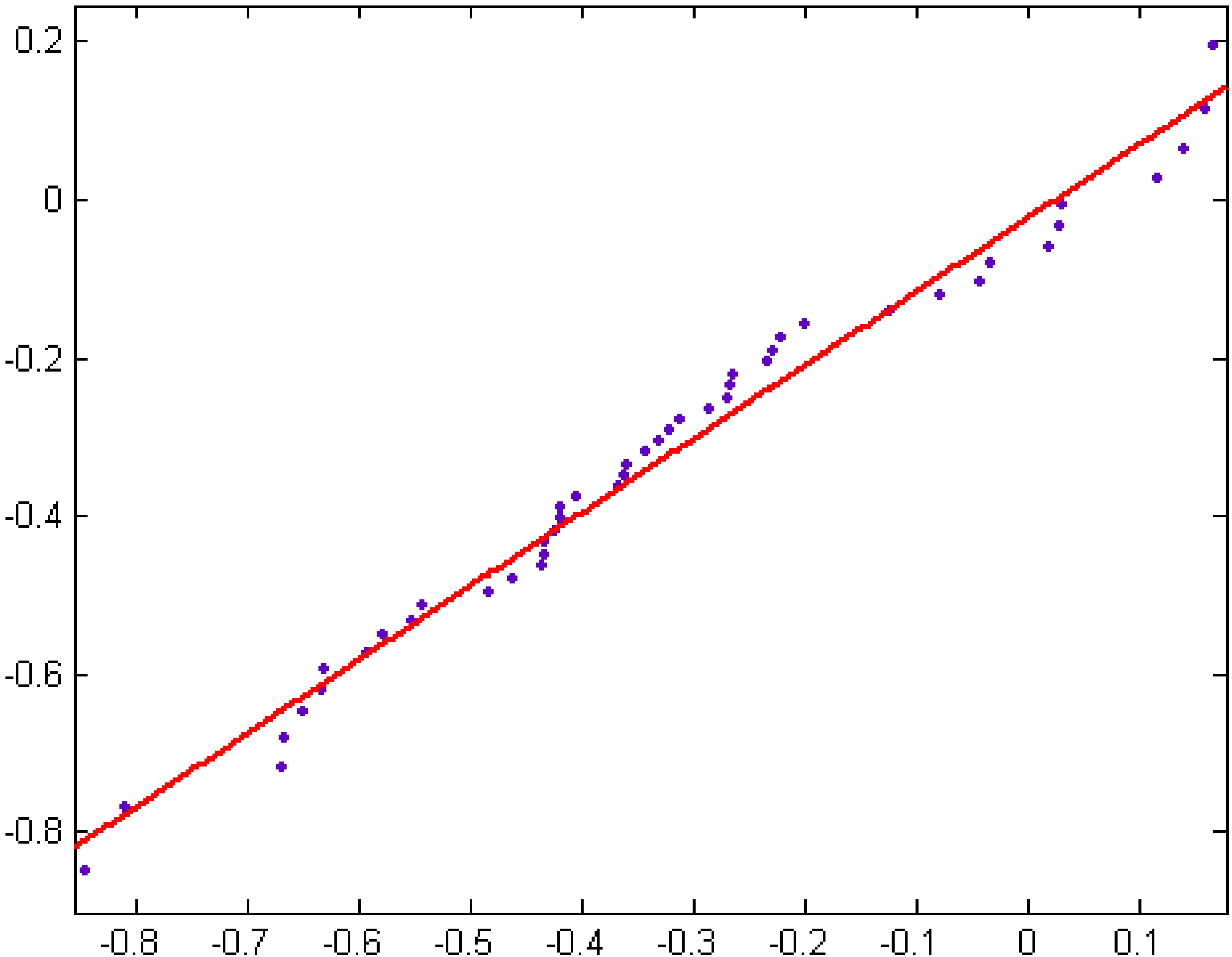


Figure 185 (n=pqrs, mean=-0.3276, standard deviation=0.2574, size=46)

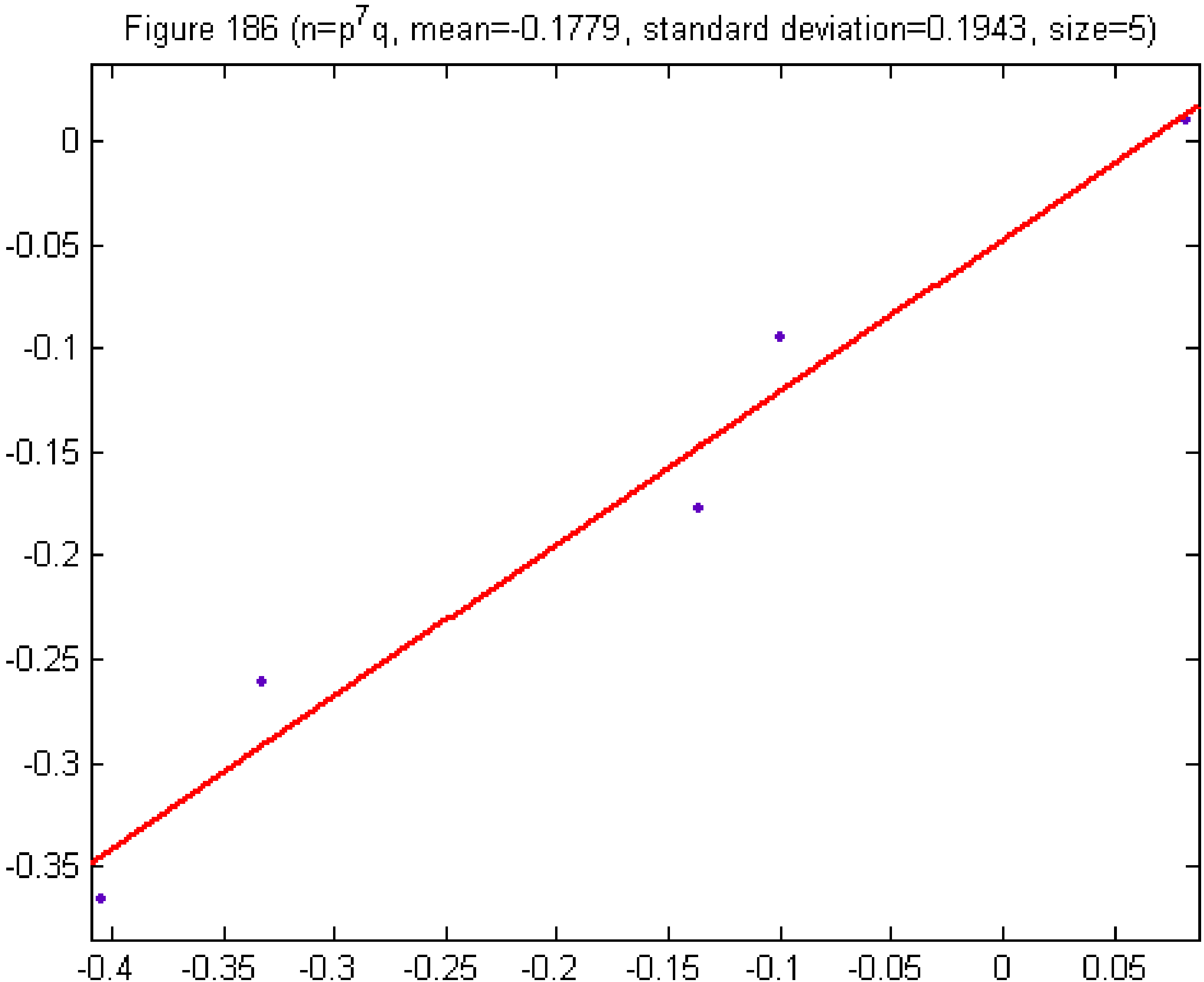


Figure 186 ($n=p^7q$, mean=-0.1779, standard deviation=0.1943, size=5)
0
-0.05
-0.1
-0.15
-0.2
-0.25
-0.3
-0.35
-0.4
-0.35
-0.3
-0.25
-0.2
-0.15
-0.1
-0.05
0
0.05

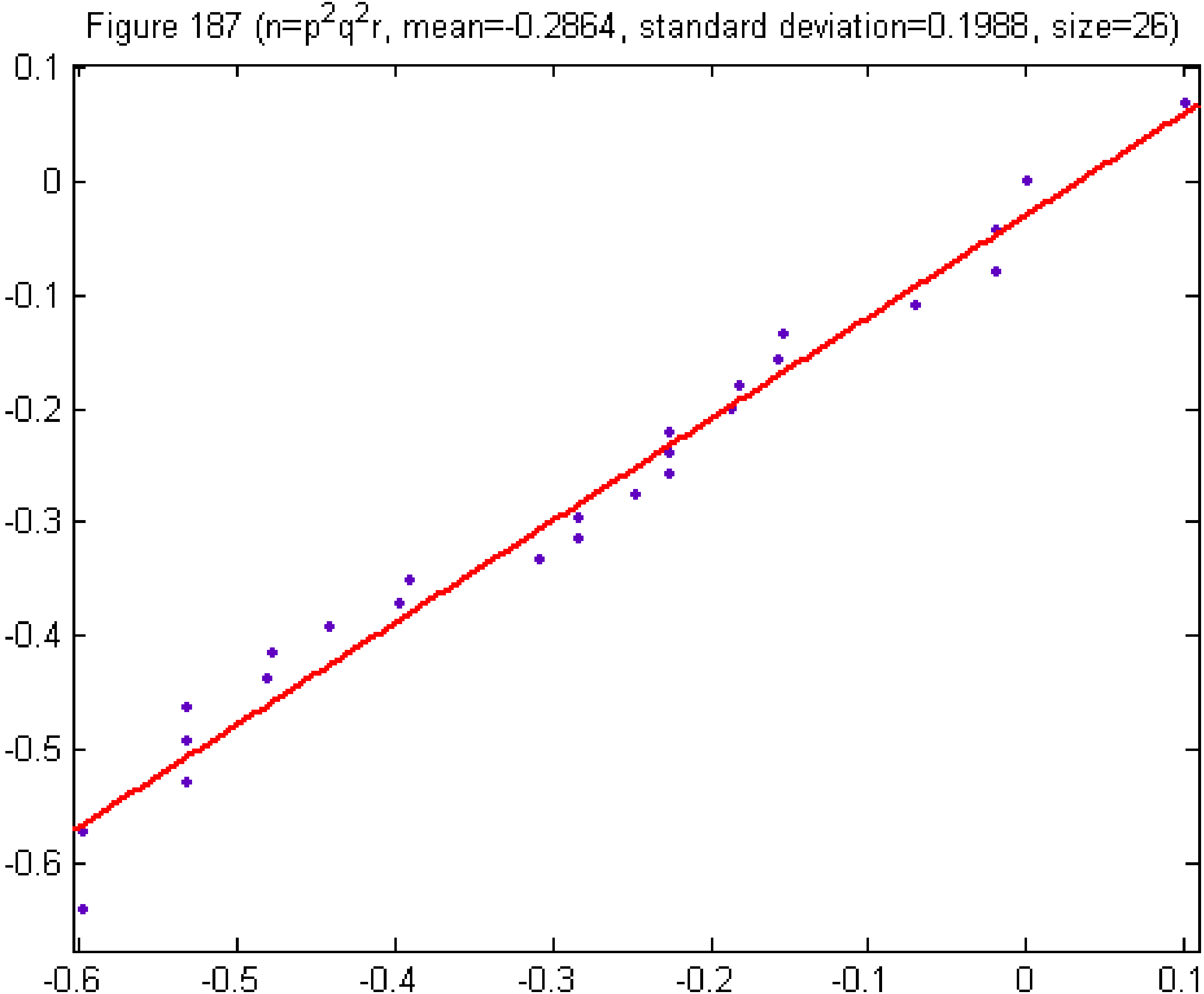


Figure 187 (n=p2q2r, mean=-0.2864, standard deviation=0.1988, size=26)
0.1
0
-0.1
-0.2
-0.3
-0.4
-0.5
-0.6
-0.6
-0.5
-0.4
-0.3
-0.2
-0.1
0
0.1

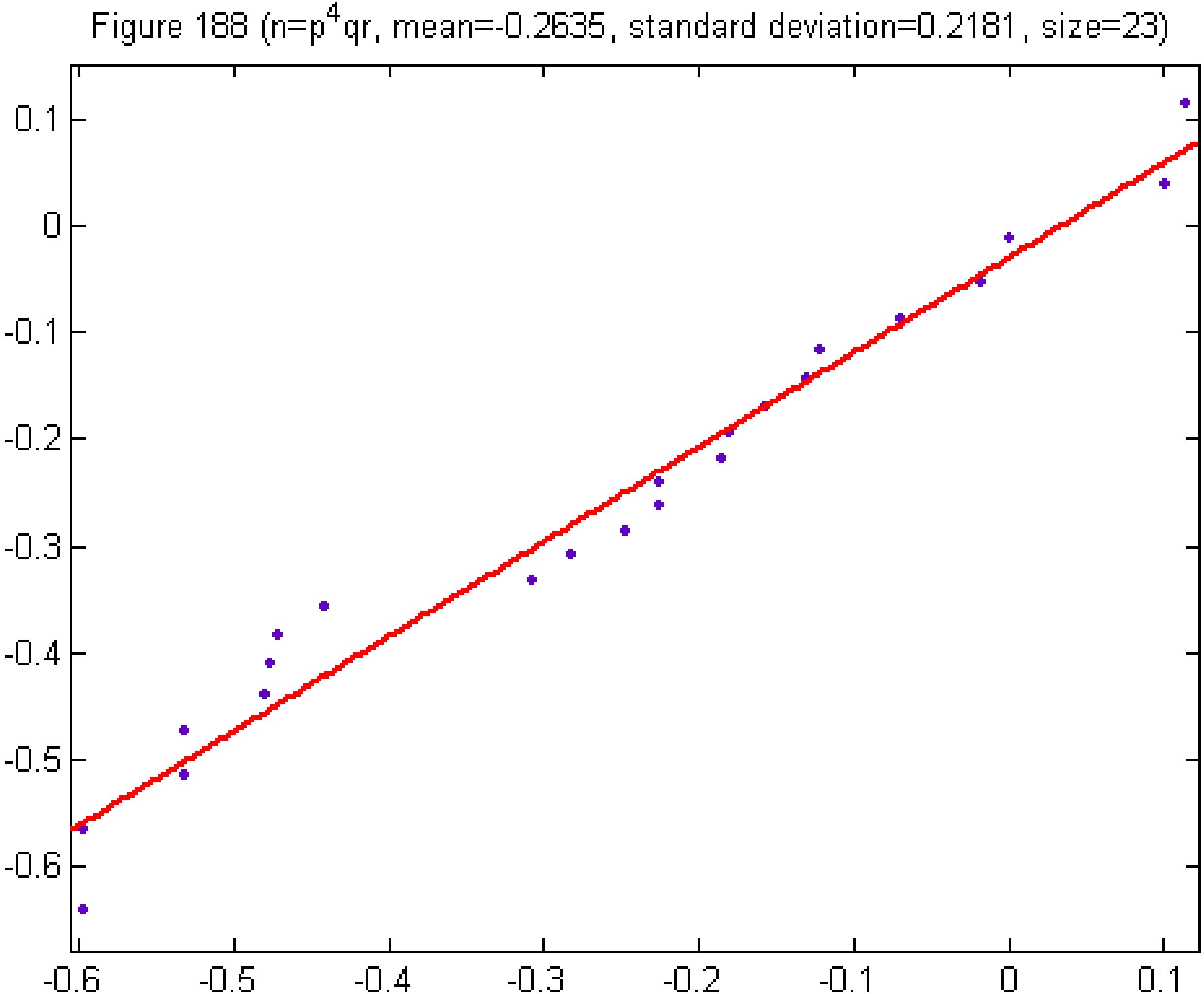


Figure 188 (n=p4qr, mean=-0.2635, standard deviation=0.2181, size=23)
0.1
0
-0.1
-0.2
-0.3
-0.4
-0.5
-0.6
-0.6
-0.5
-0.4
-0.3
-0.2
-0.1
0
0.1

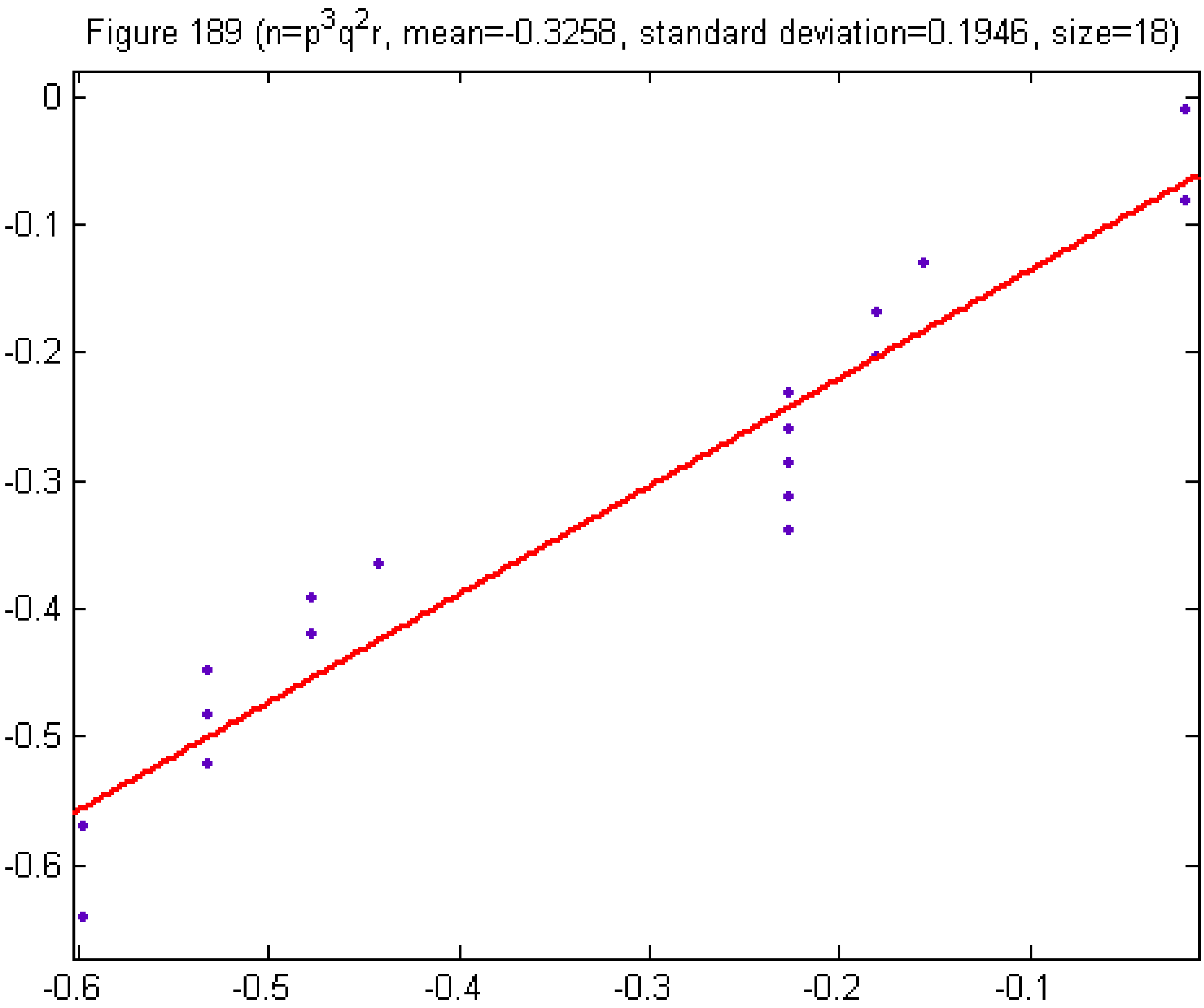


Figure 189 ($n=p^3q^2r$, mean=-0.3258, standard deviation=0.1946, size=18)
0
-0.1
-0.2
-0.3
-0.4
-0.5
-0.6
-0.6
-0.5
-0.4
-0.3
-0.2
-0.1

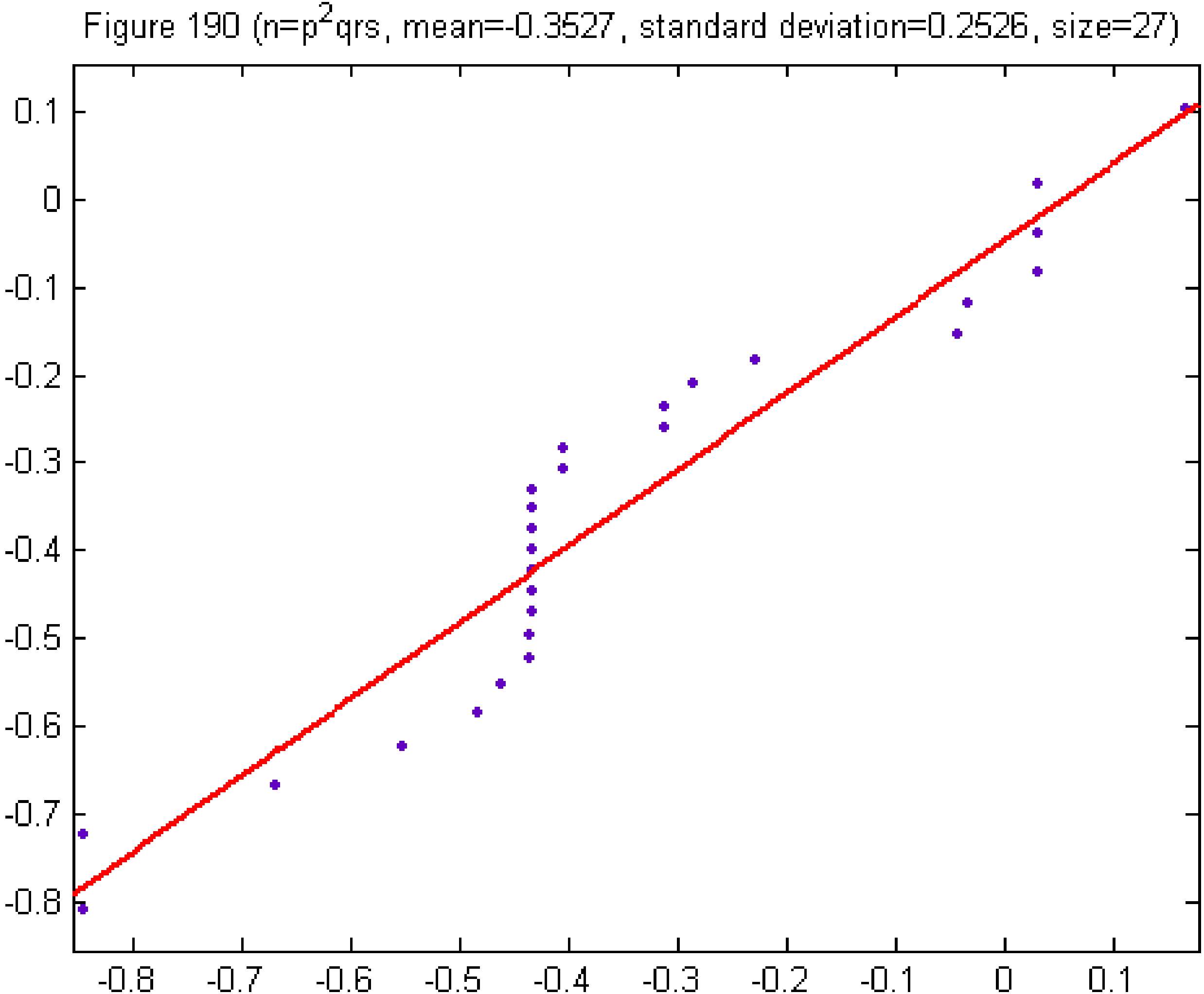


Figure 190 (n=p2qrs, mean=-0.3527, standard deviation=0.2526, size=27)
0.1
0
-0.1
-0.2
-0.3
-0.4
-0.5
-0.6
-0.7
-0.8
-0.8
-0.7
-0.6
-0.5
-0.4
-0.3
-0.2
-0.1
0
0.1

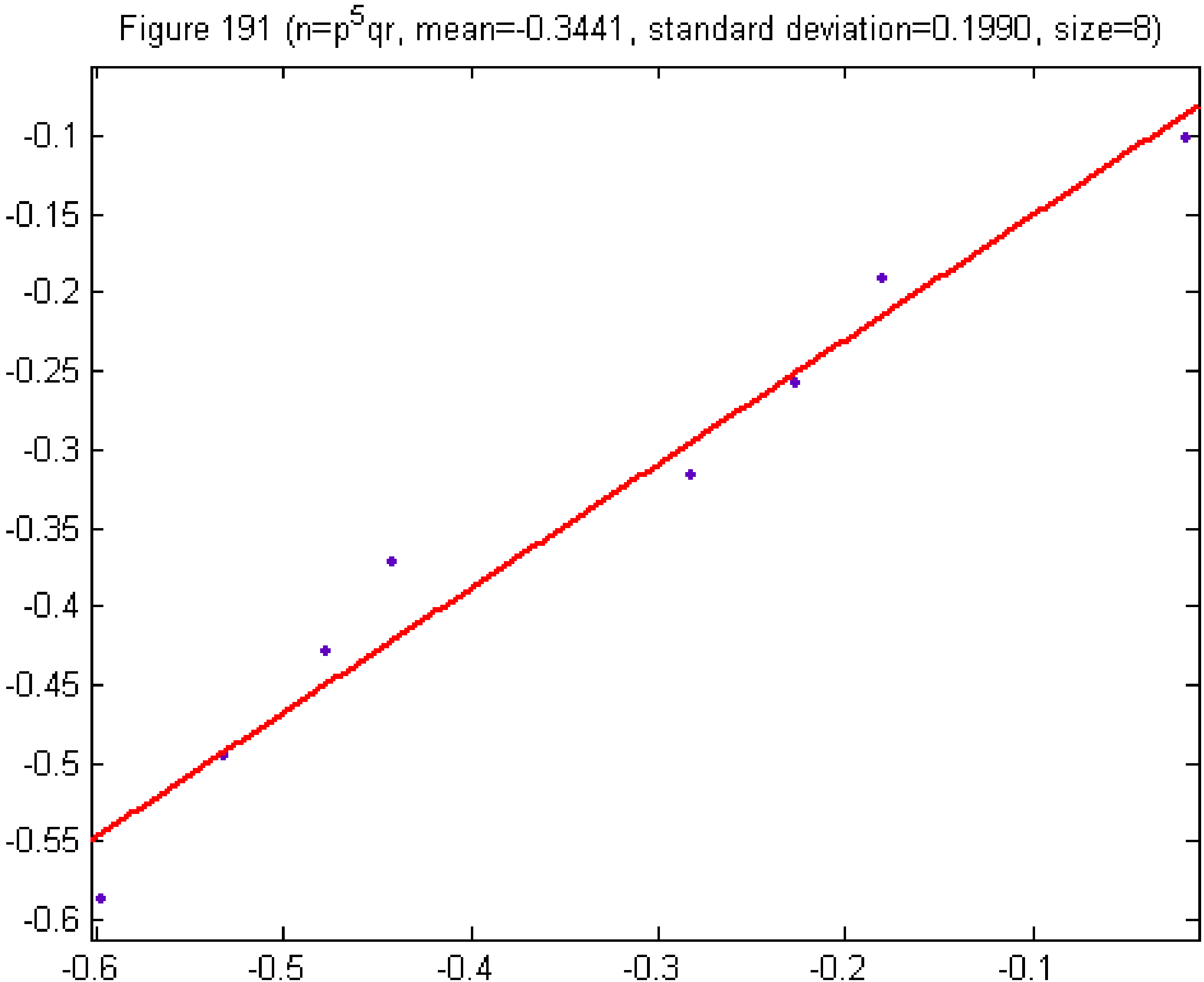


Figure 191 (n=p5qr, mean=-0.3441, standard deviation=0.1990, size=8)
-0.1
-0.15
-0.2
-0.25
-0.3
-0.35
-0.4
-0.45
-0.5
-0.55
-0.6
-0.6
-0.5
-0.4
-0.3
-0.2
-0.1

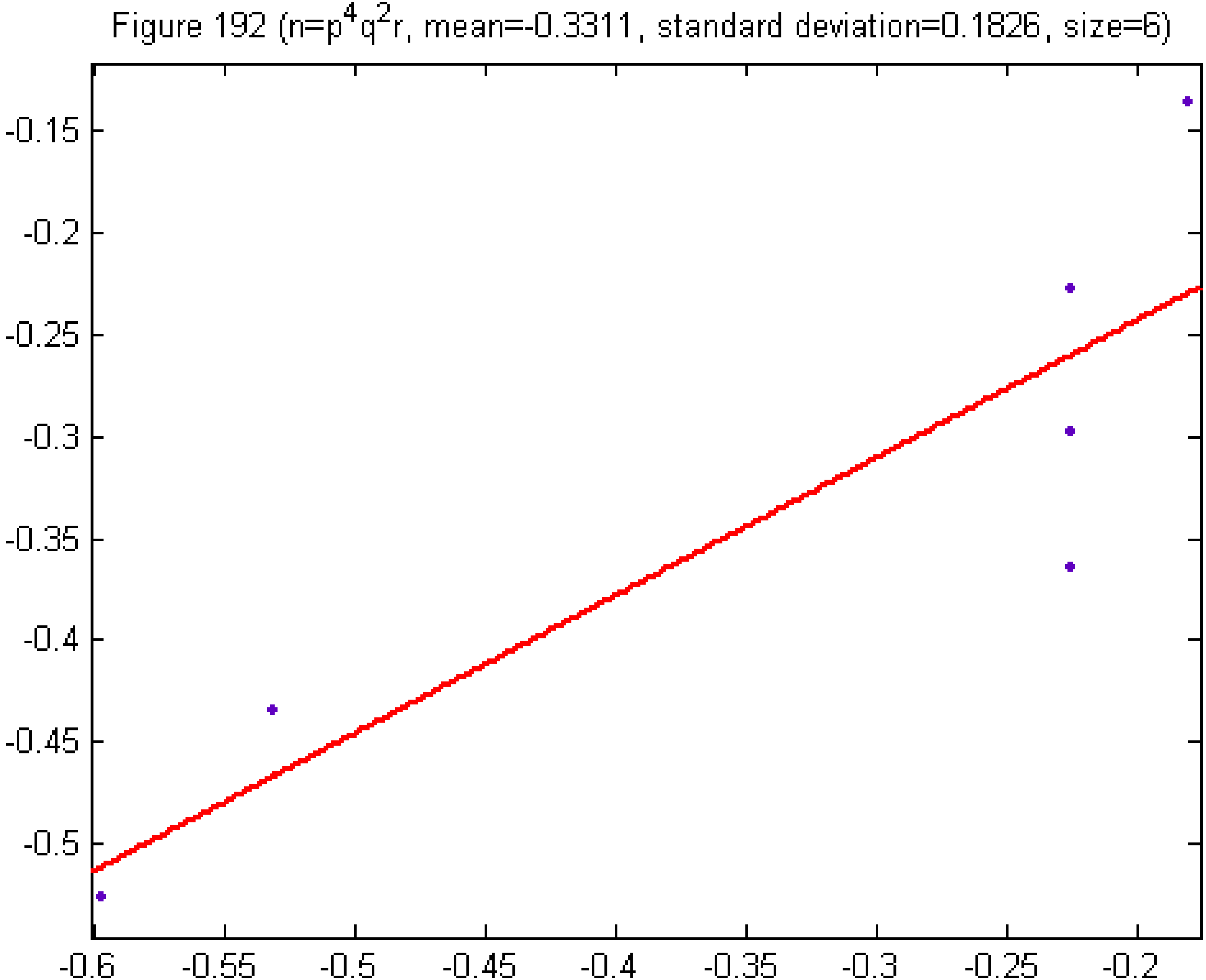
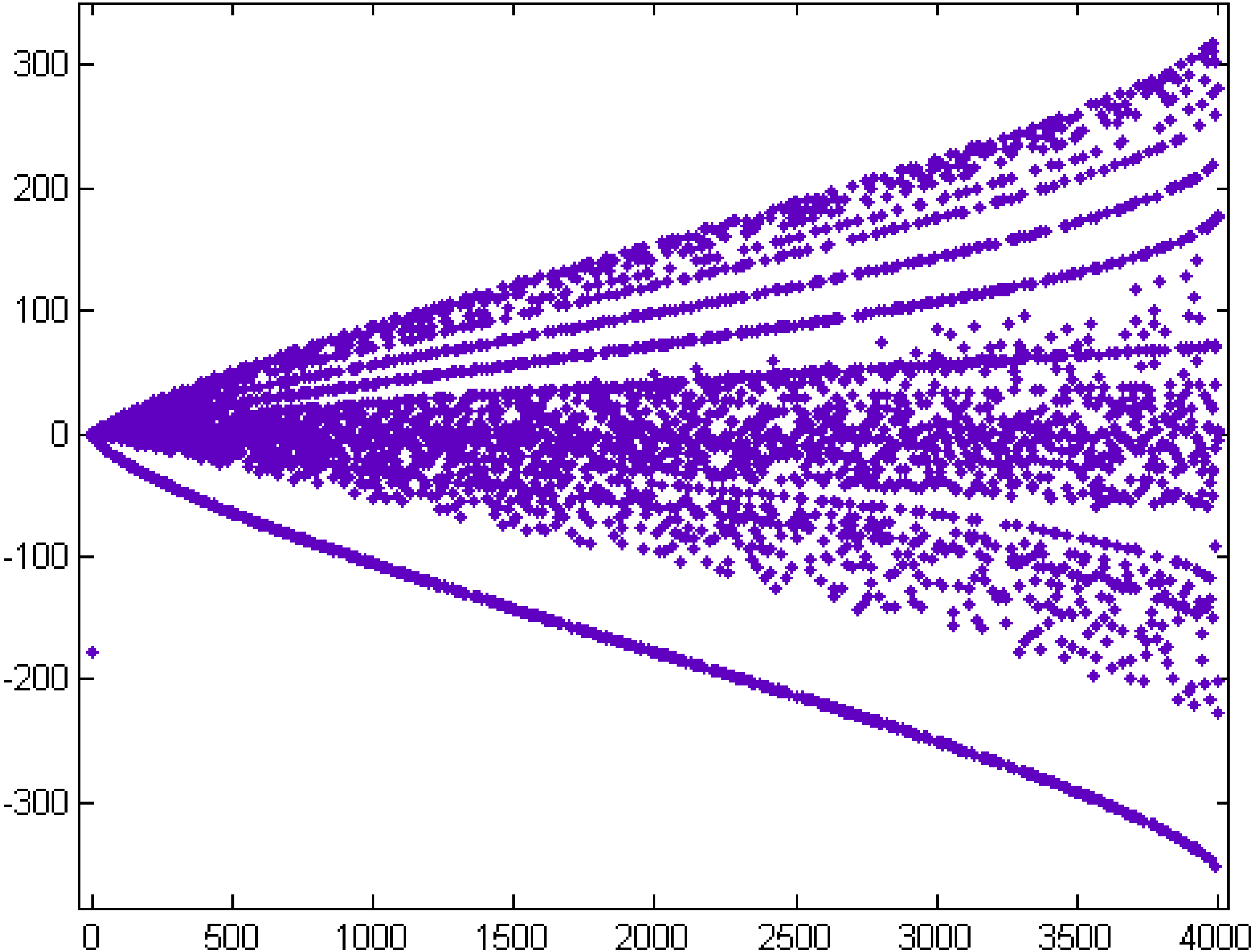


Figure 192 (n=p4q2r, mean=-0.3311, standard deviation=0.1826, size=6)
-0.15
-0.2
-0.25
-0.3
-0.35
-0.4
-0.45
-0.5
-0.6
-0.55
-0.5
-0.45
-0.4
-0.35
-0.3
-0.25
-0.2

Figure 193



300
200
100
0
-100
-200
-300
0
500
1000
1500
2000
2500
3000
3500
4000

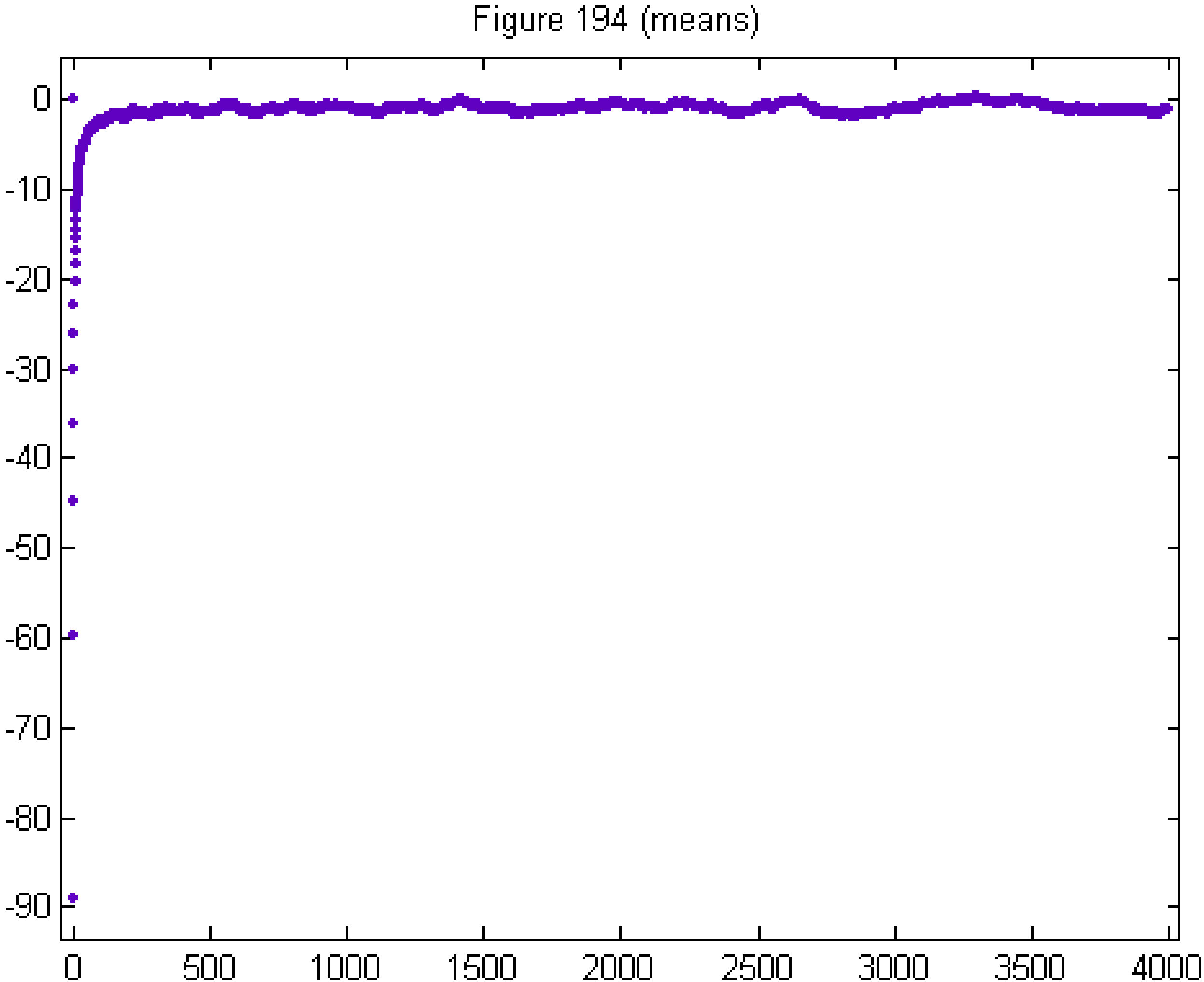


Figure 194 (means)
0
-10
-20
-30
-40
-50
-60
-70
-80
-90
0
500
1000
1500
2000
2500
3000
3500
4000

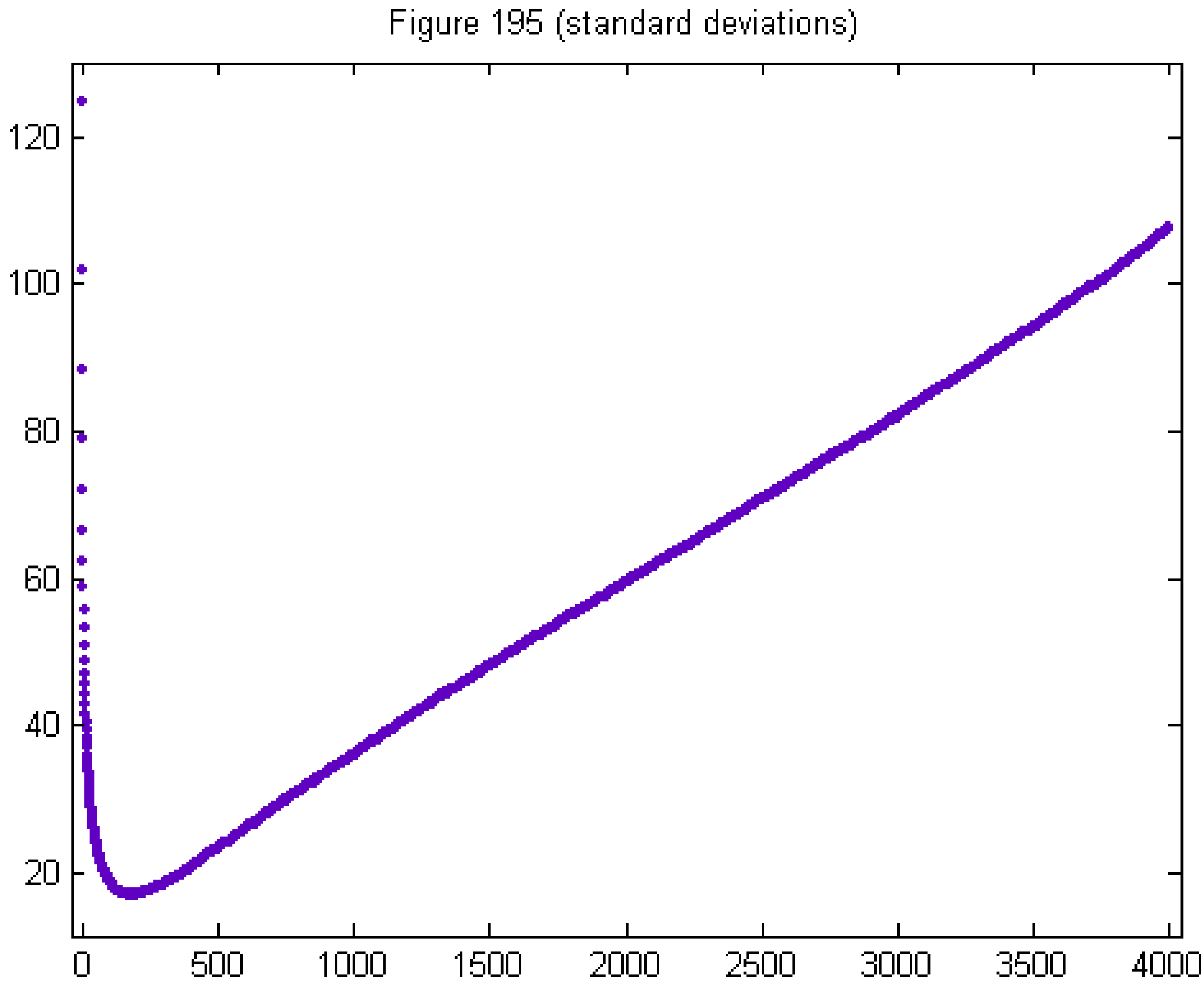
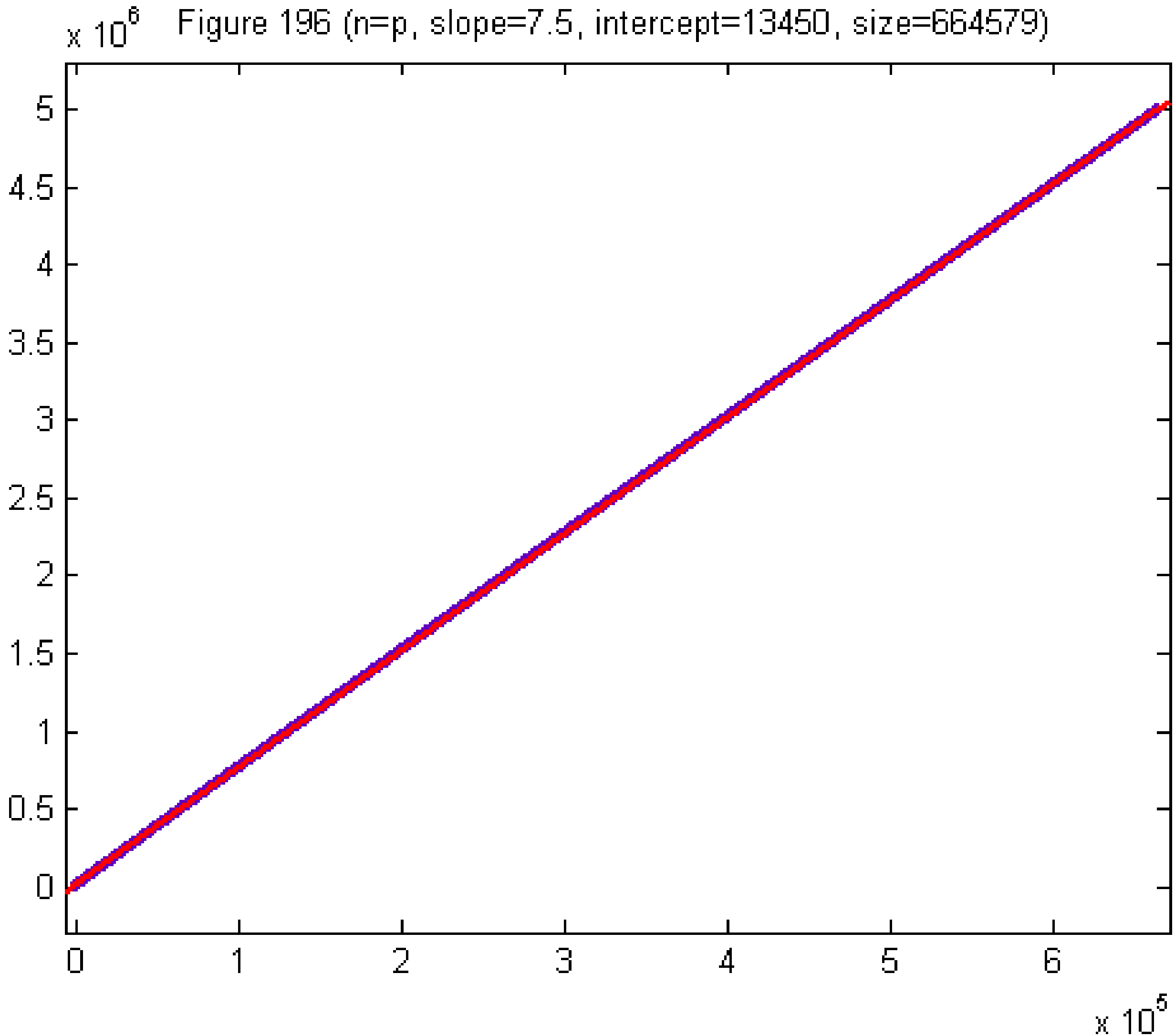
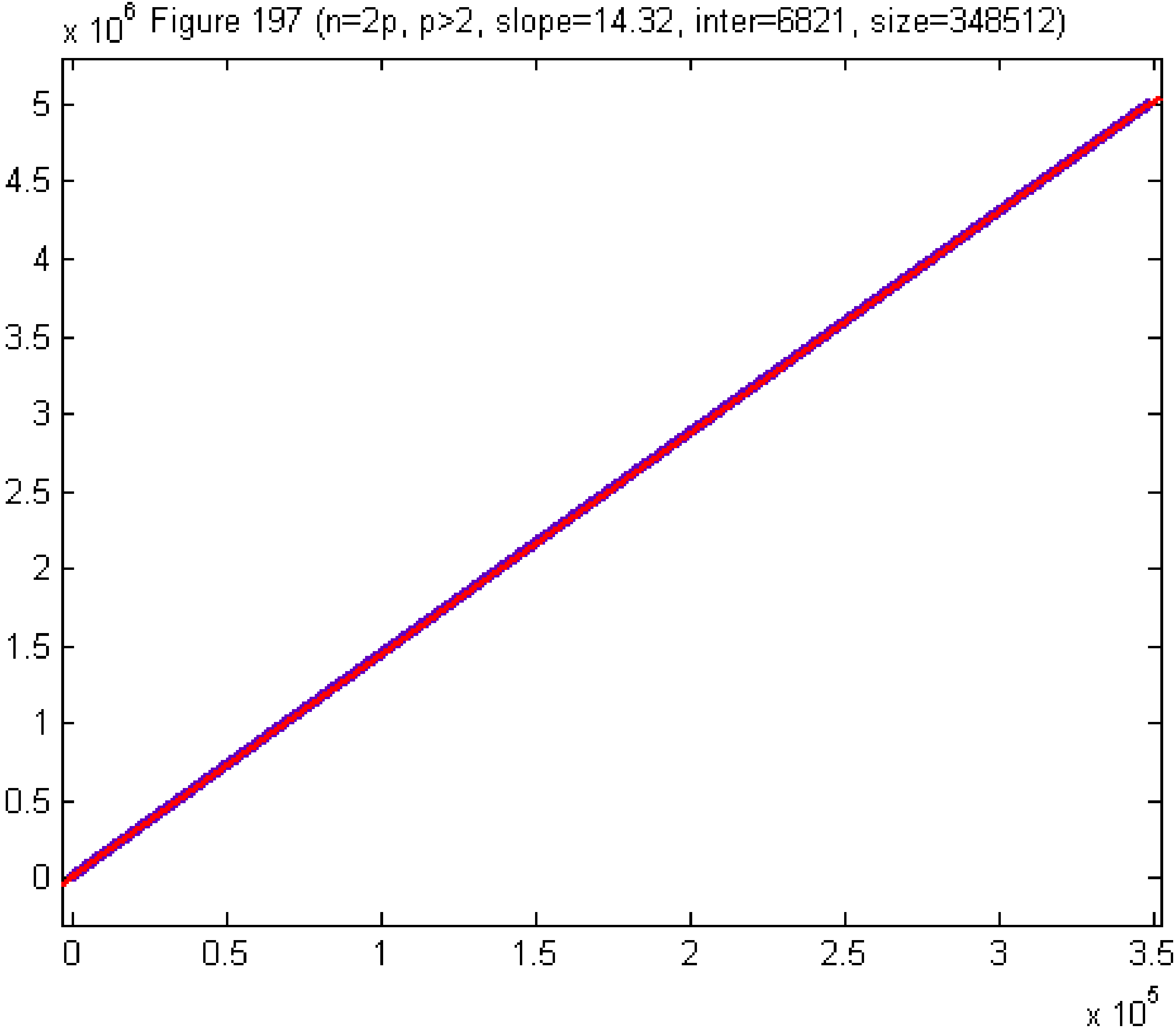


Figure 195 (standard deviations)
120
100
80
60
40
20
0
500
1000
1500
2000
2500
3000
3500
4000



x 10^6
Figure 196 (n=p, slope=7.5, intercept=13450, size=664579)
5
4.5
4
3.5
3
2.5
2
1.5
1
0.5
0
0
1
2
3
4
5
6
x 10^5


x 10^6 Figure 197 (n=2p, p>2, slope=14.32, inter=6821, size=348512)
5
4.5
4
3.5
3
2.5
2
1.5
1
0.5
0
0
0.5
1
1.5
2
2.5
3
3.5
x 10^5

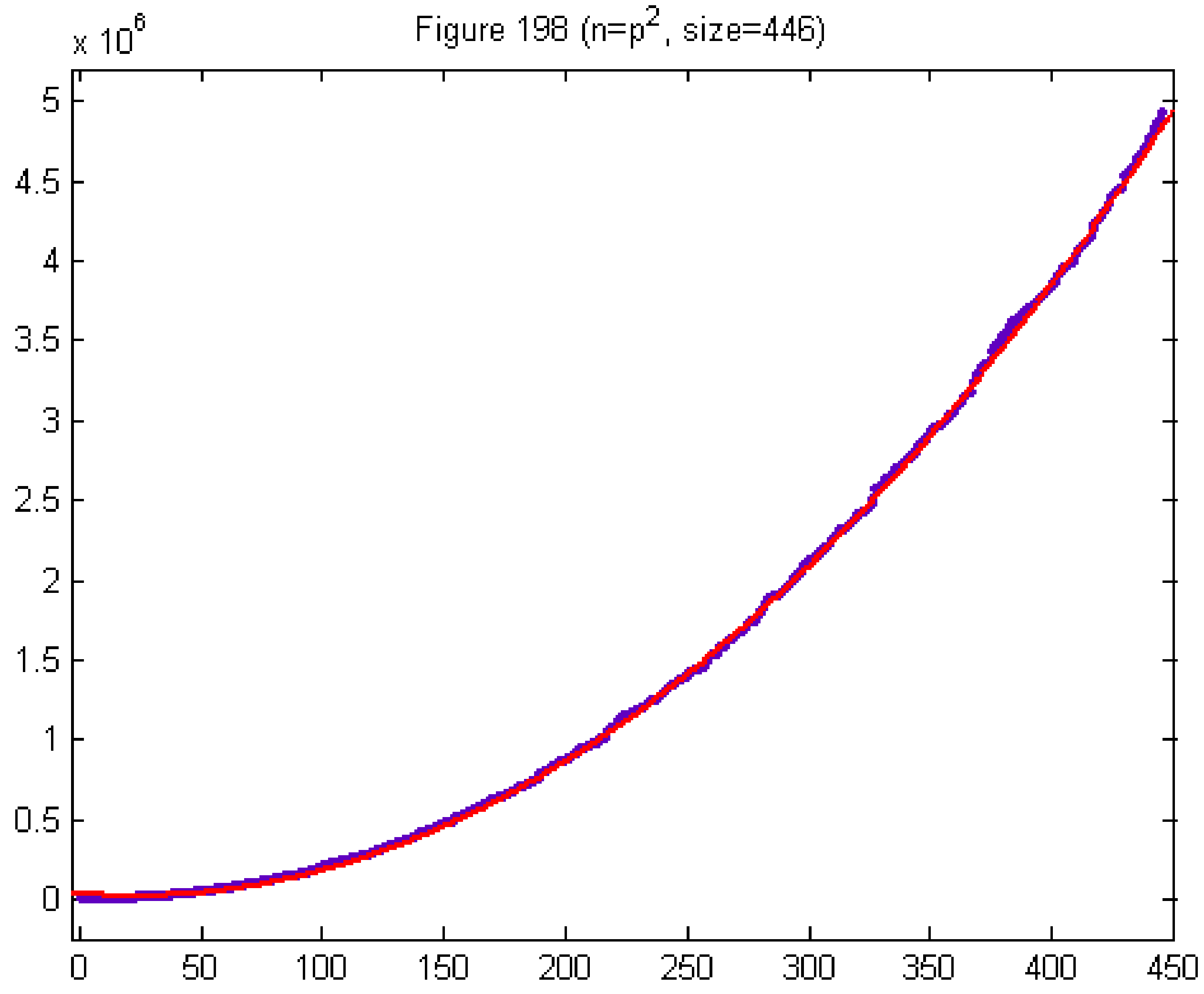


Figure 198 (n=p^2, size=446)
x 10^6
5
4.5
4
3.5
3
2.5
2
1.5
1
0.5
0
0
50
100
150
200
250
300
350
400
450

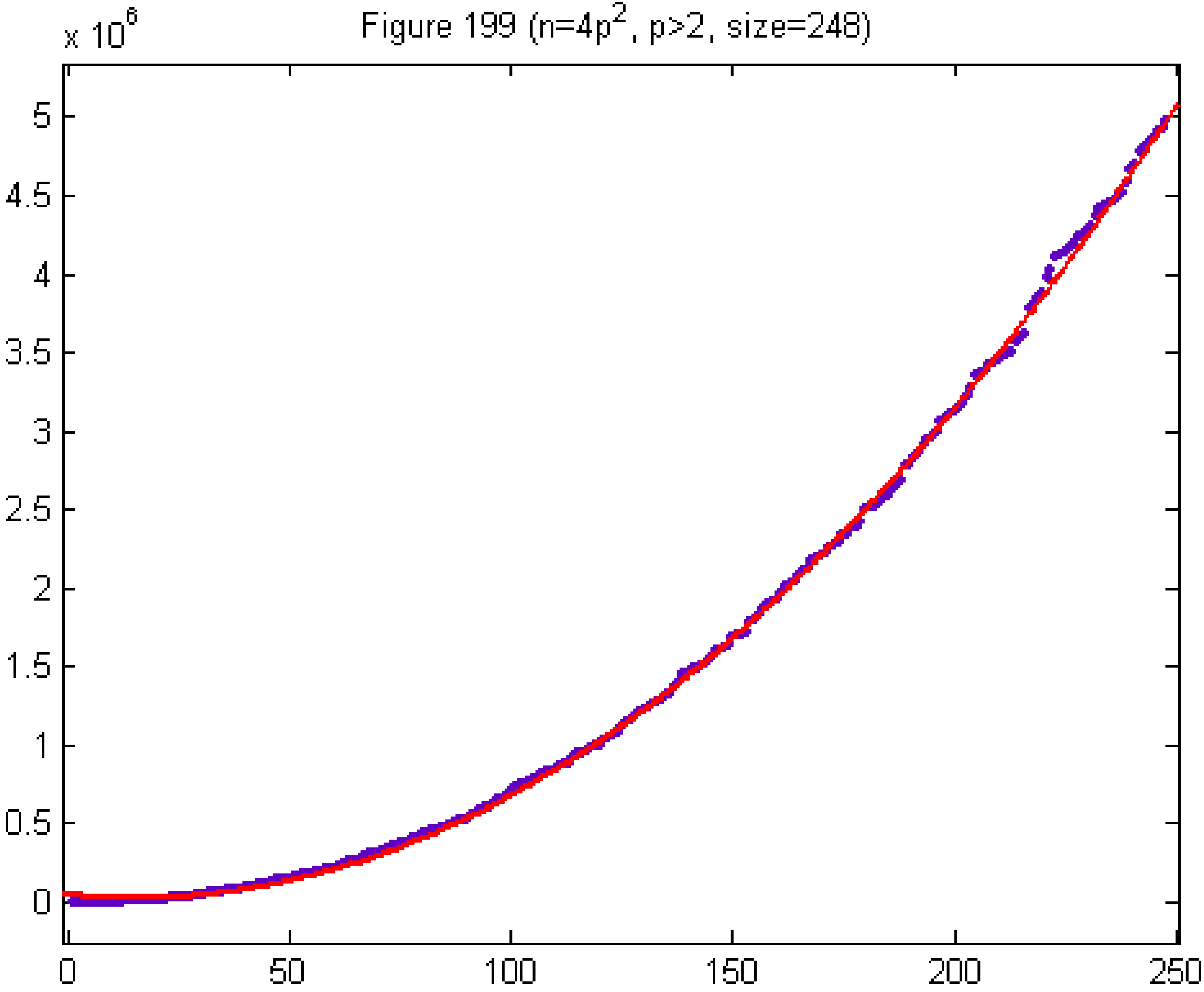


Figure 199 (n=4p2, p>2, size=248)
x 10^6
5
4.5
4
3.5
3
2.5
2
1.5
1
0.5
0
0
50
100
150
200
250

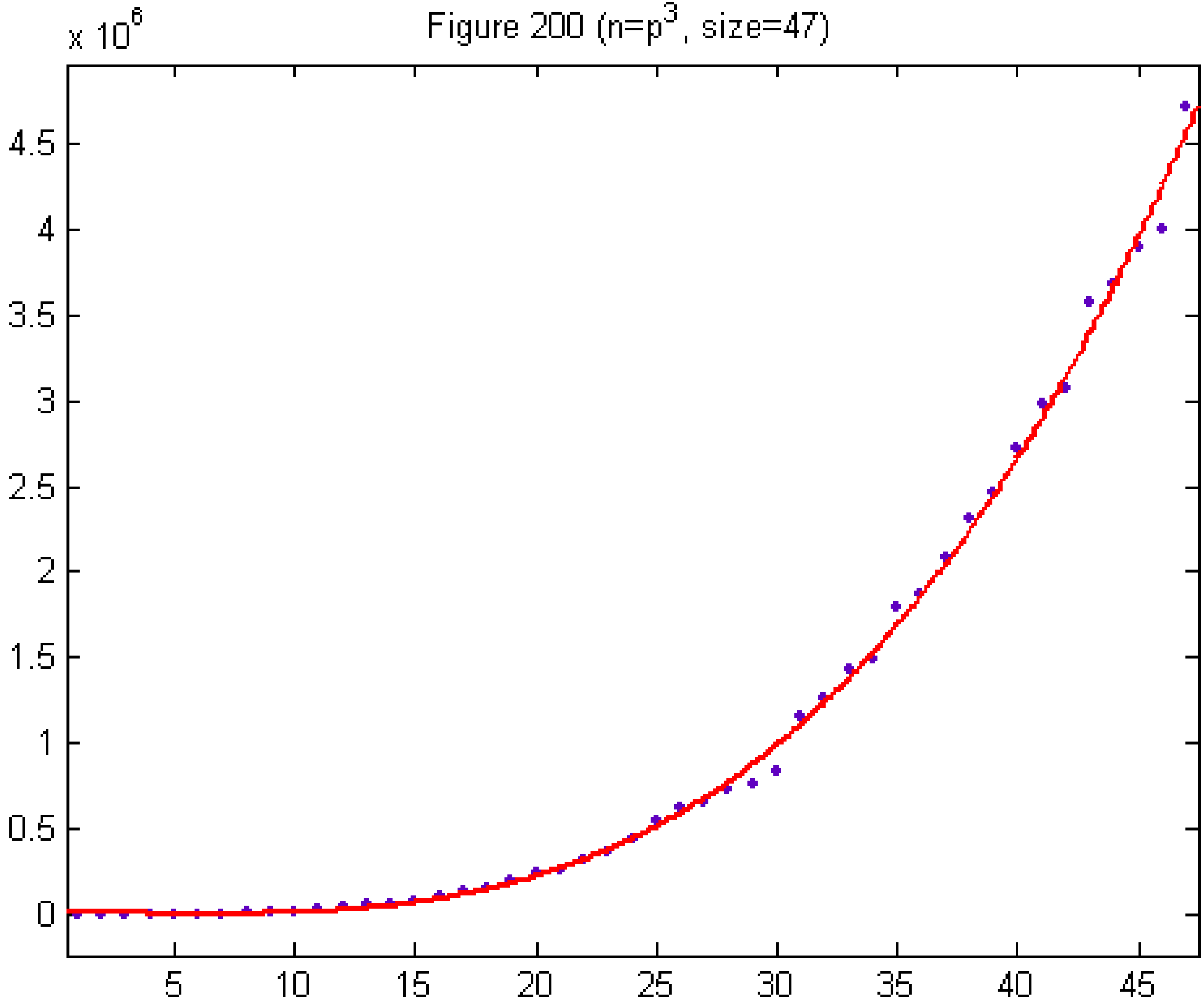
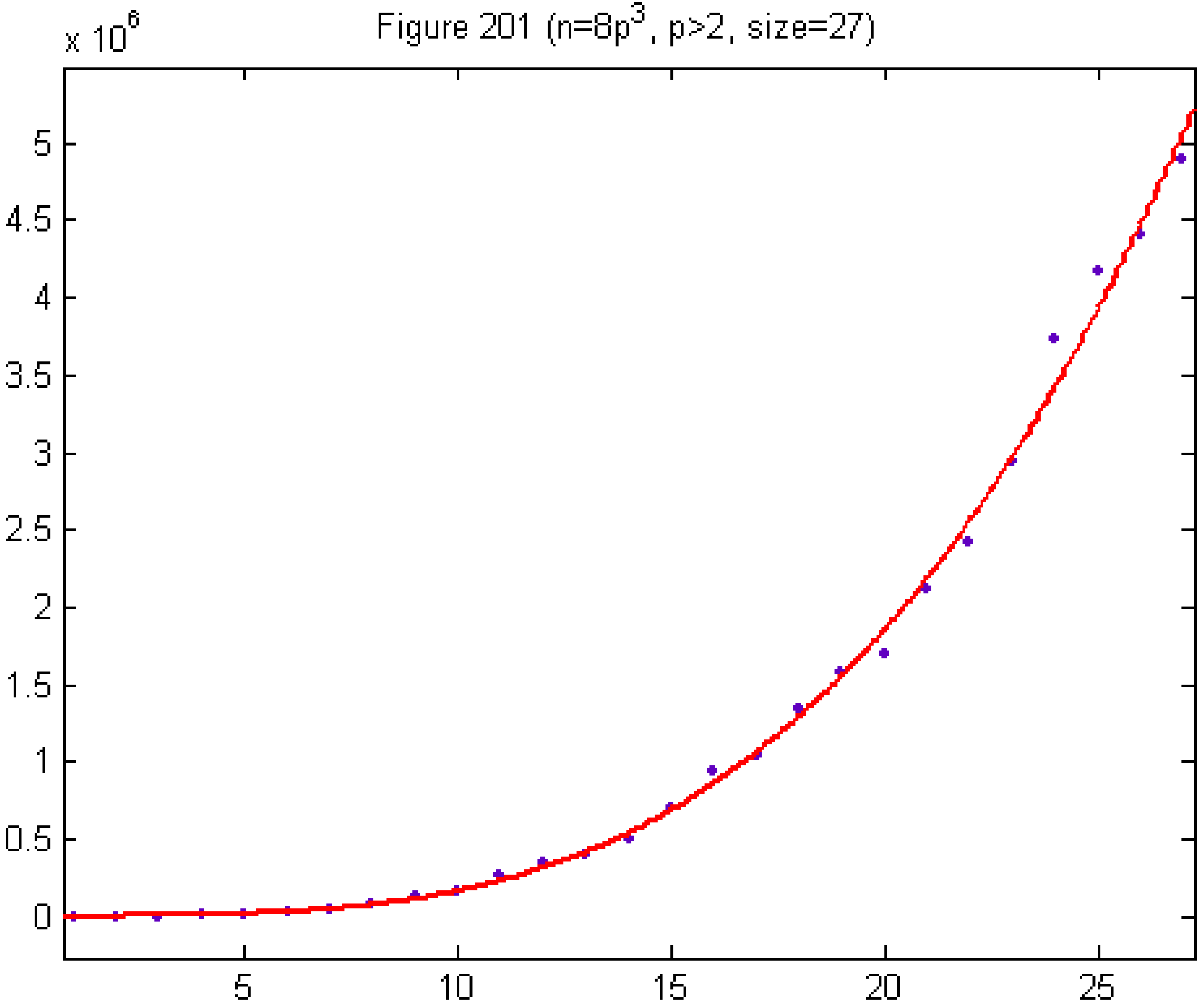


Figure 200 (n=p^3, size=47)
x 10^6
4.5
4
3.5
3
2.5
2
1.5
1
0.5
0
5
10
15
20
25
30
35
40
45

Figure 201 ($n=8p^3$, $p>2$, size=27)



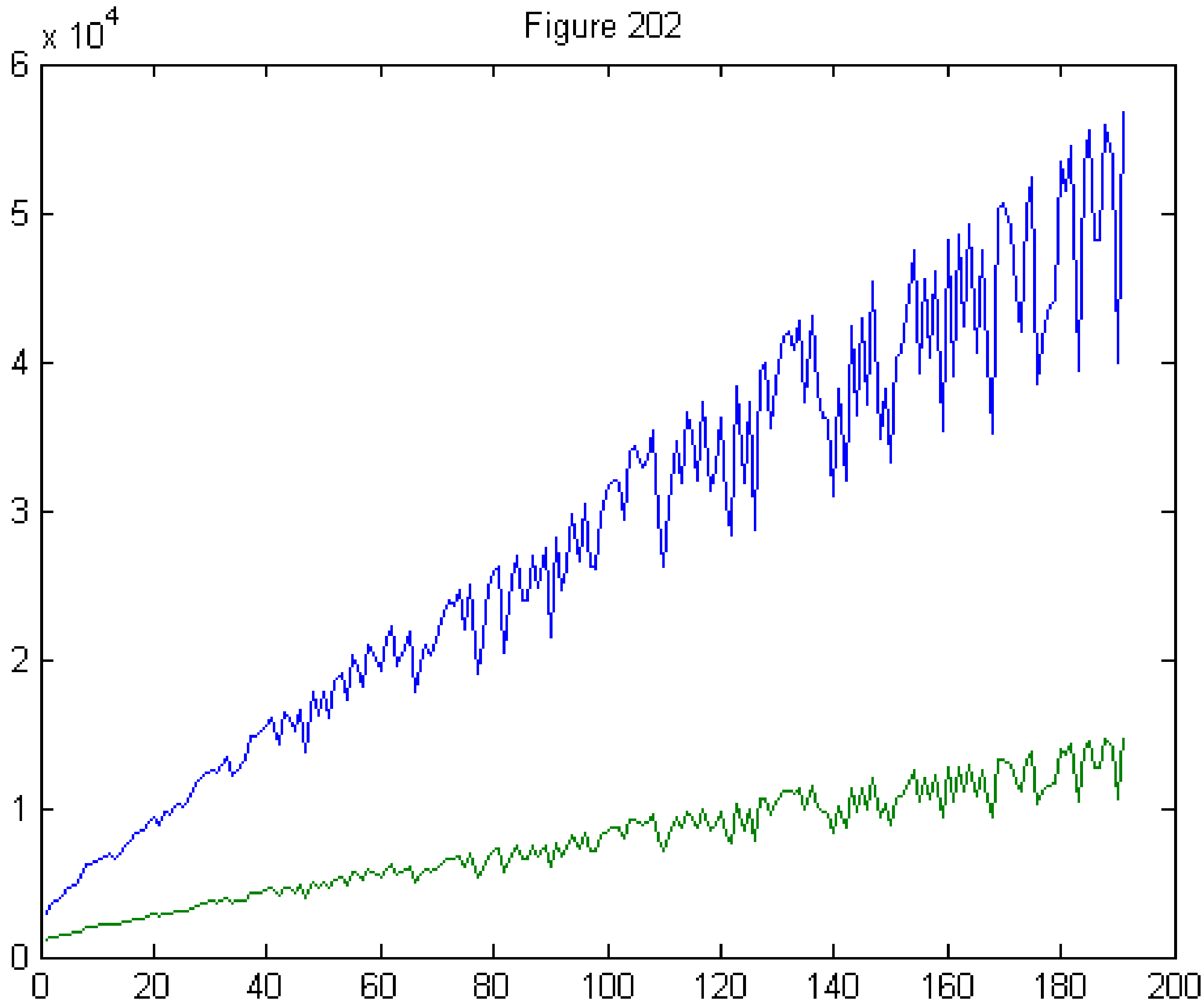


Figure 202
x 10^4
6
5
4
3
2
1
0
0
20
40
60
80
100
120
140
160
180
200

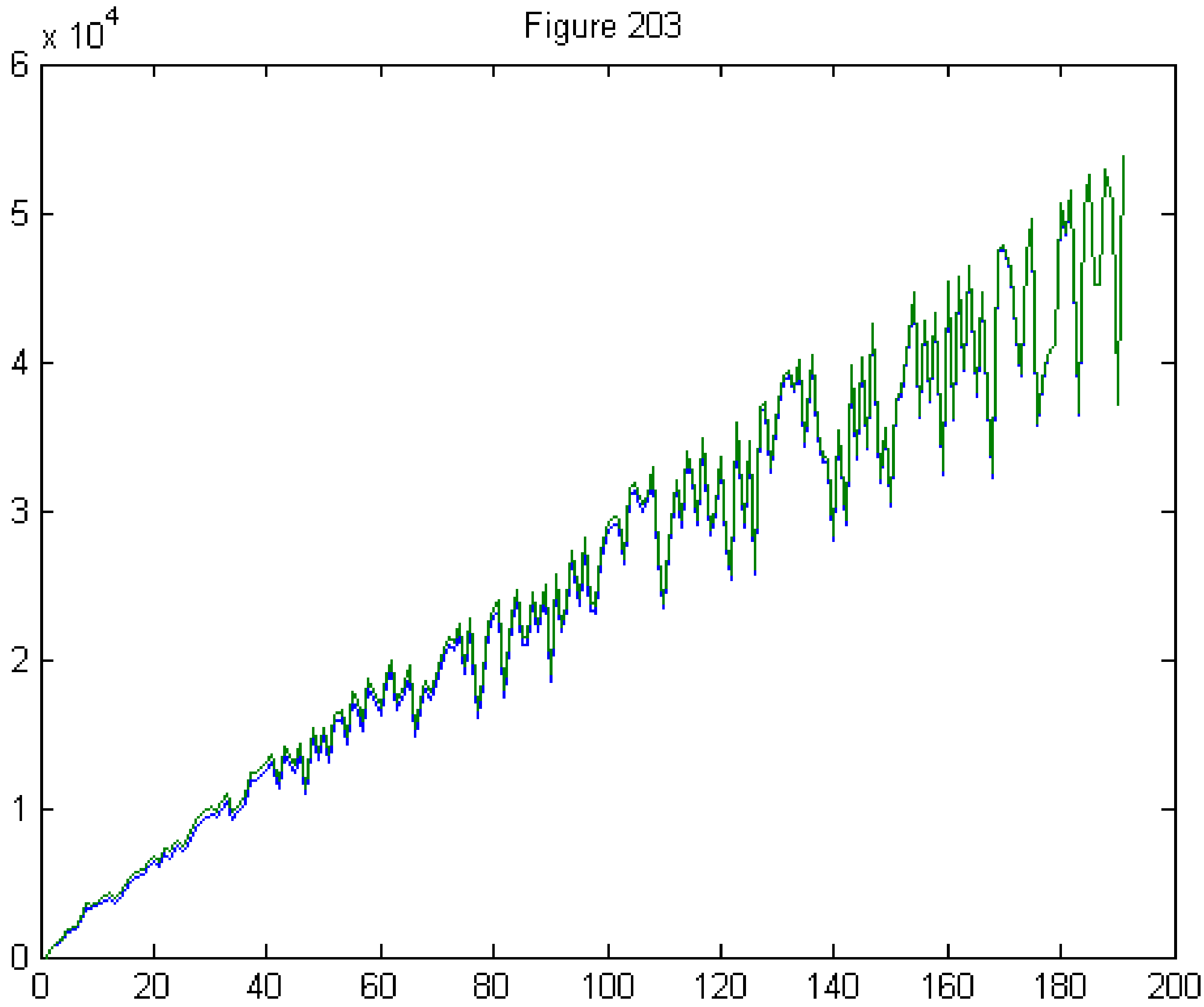


Figure 203
x 10^4
6
5
4
3
2
1
0
0
20
40
60
80
100
120
140
160
180
200

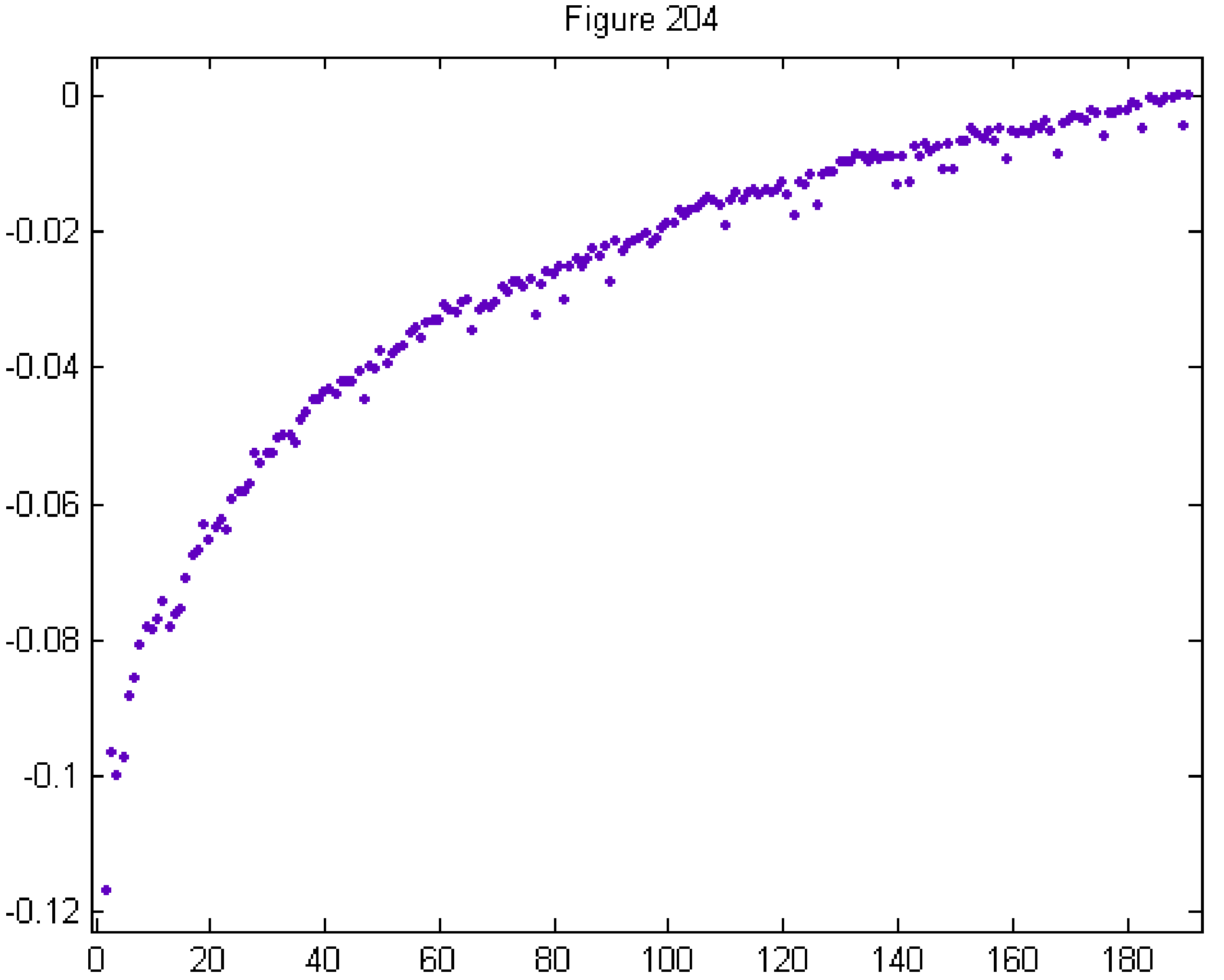
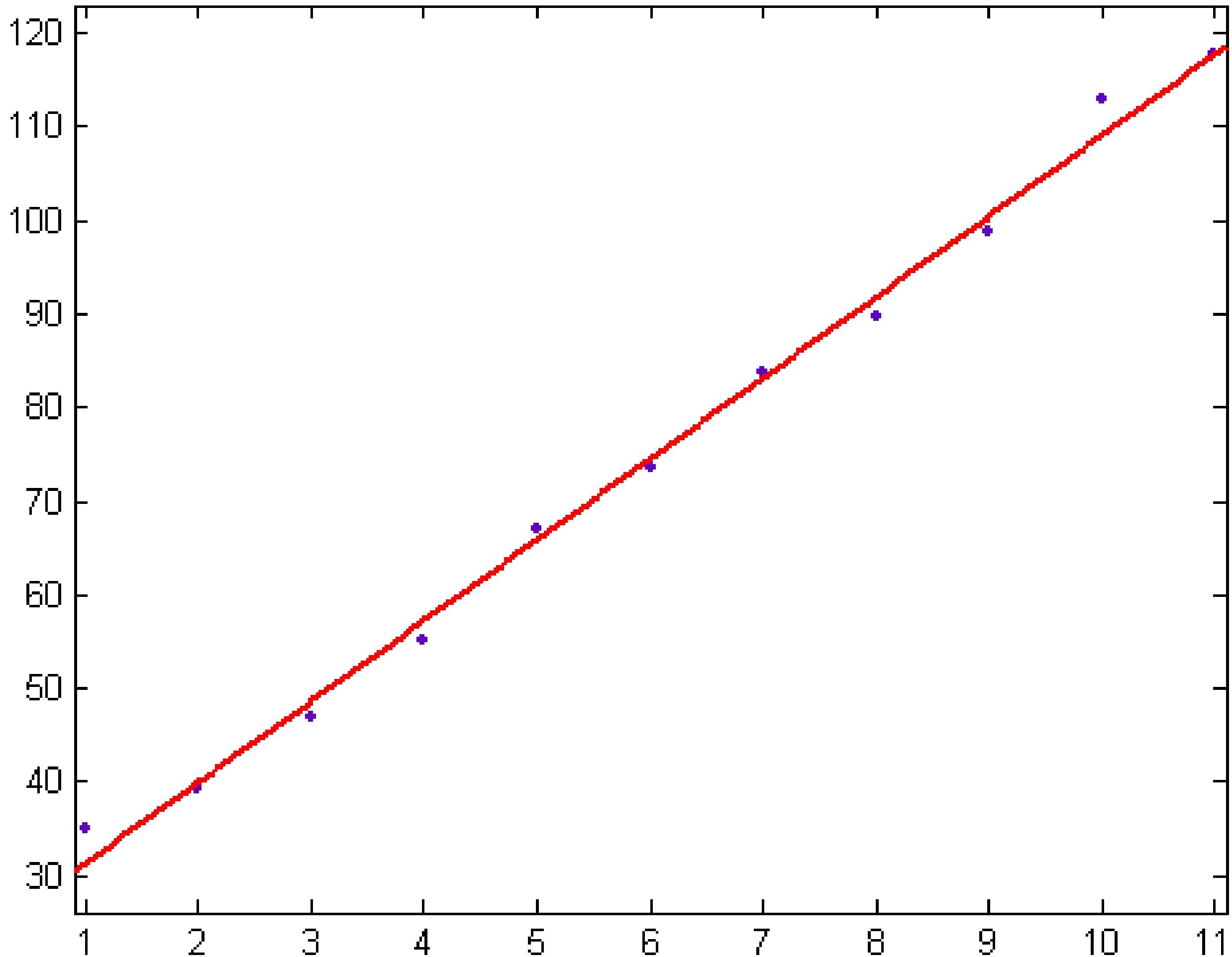


Figure 204
0
-0.02
-0.04
-0.06
-0.08
-0.1
-0.12
0
20
40
60
80
100
120
140
160
180

Figure 205



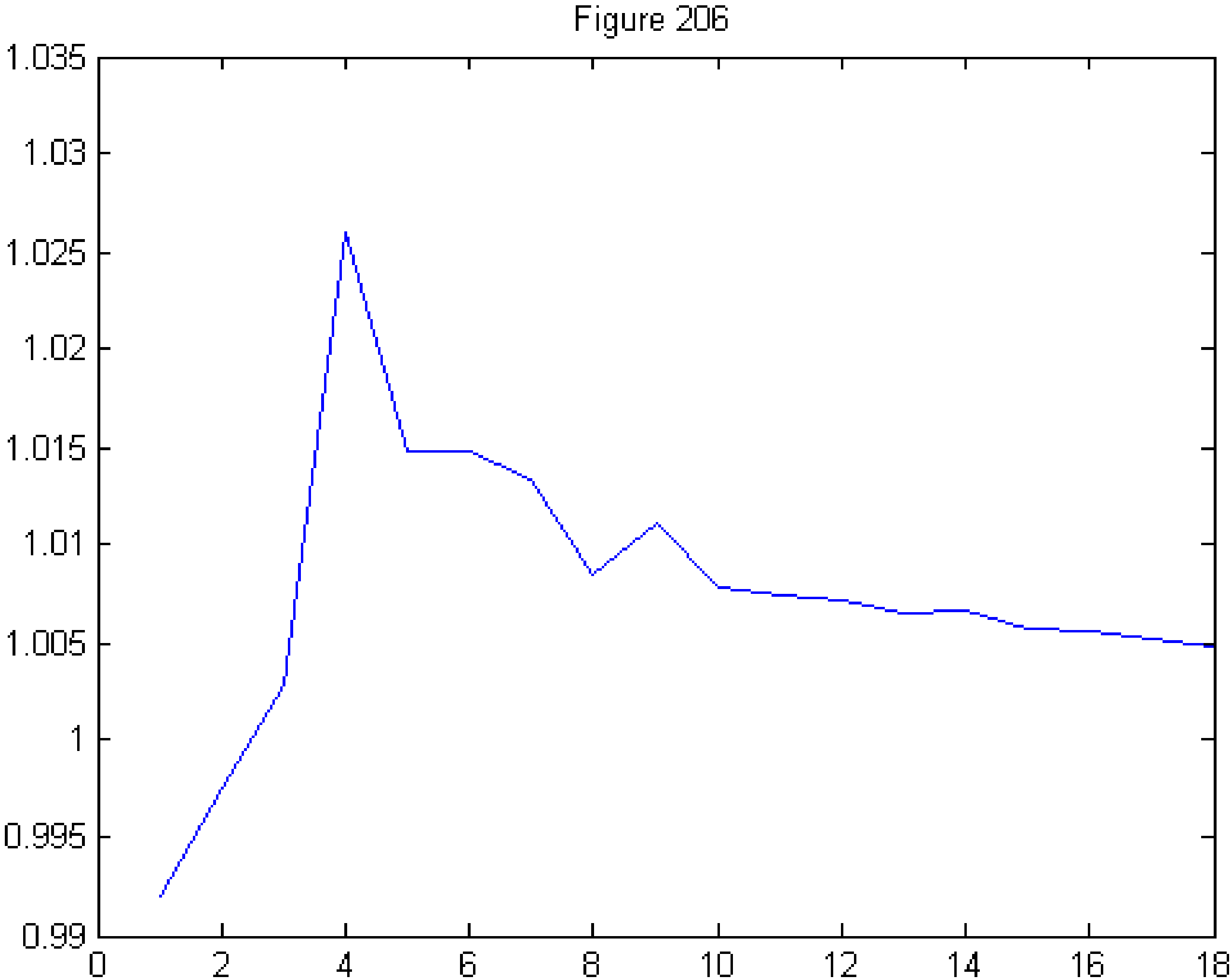
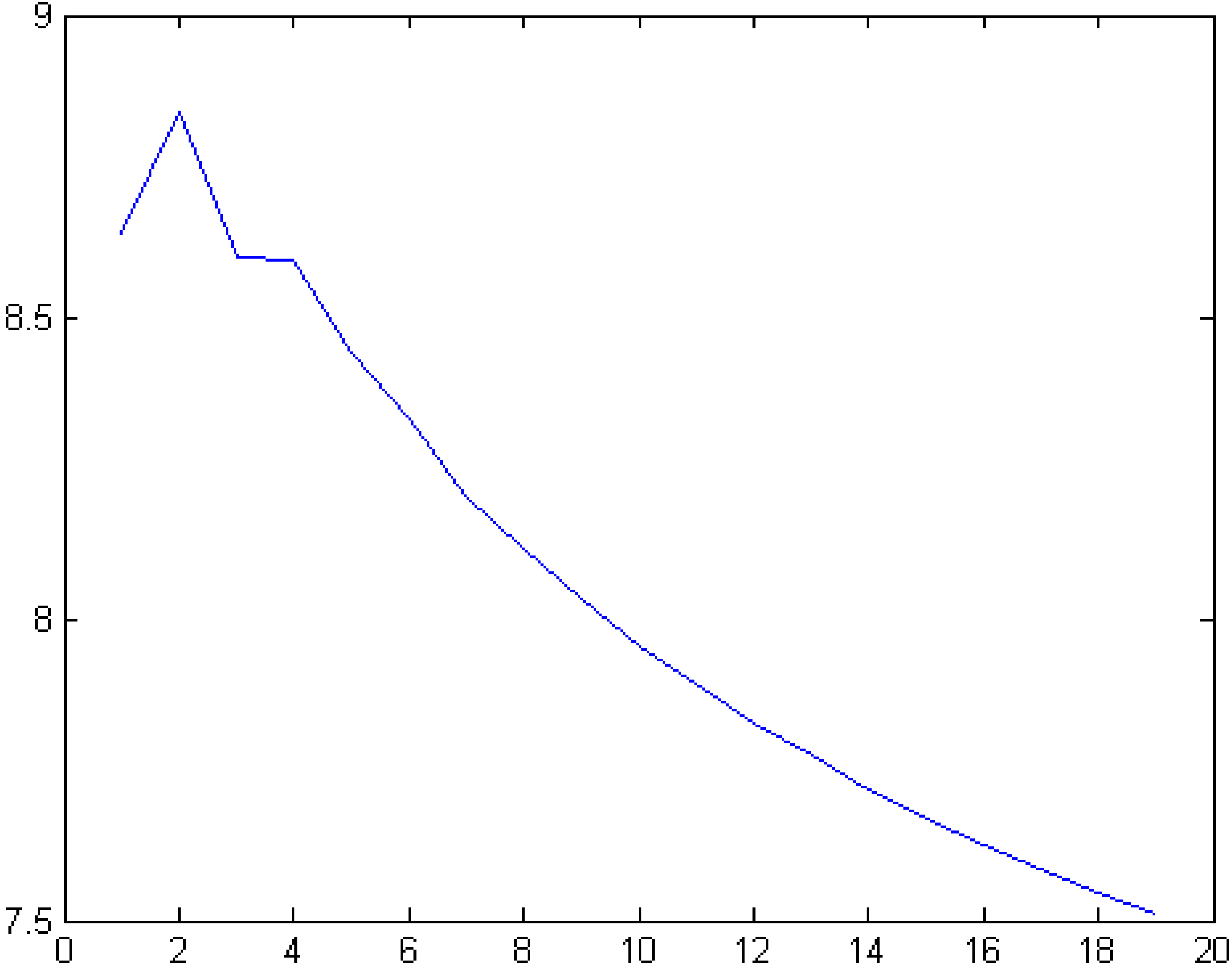


Figure 206
1.035
1.03
1.025
1.02
1.015
1.01
1.005
1
0.995
0.99
0
2
4
6
8
10
12
14
16
18

Figure 207



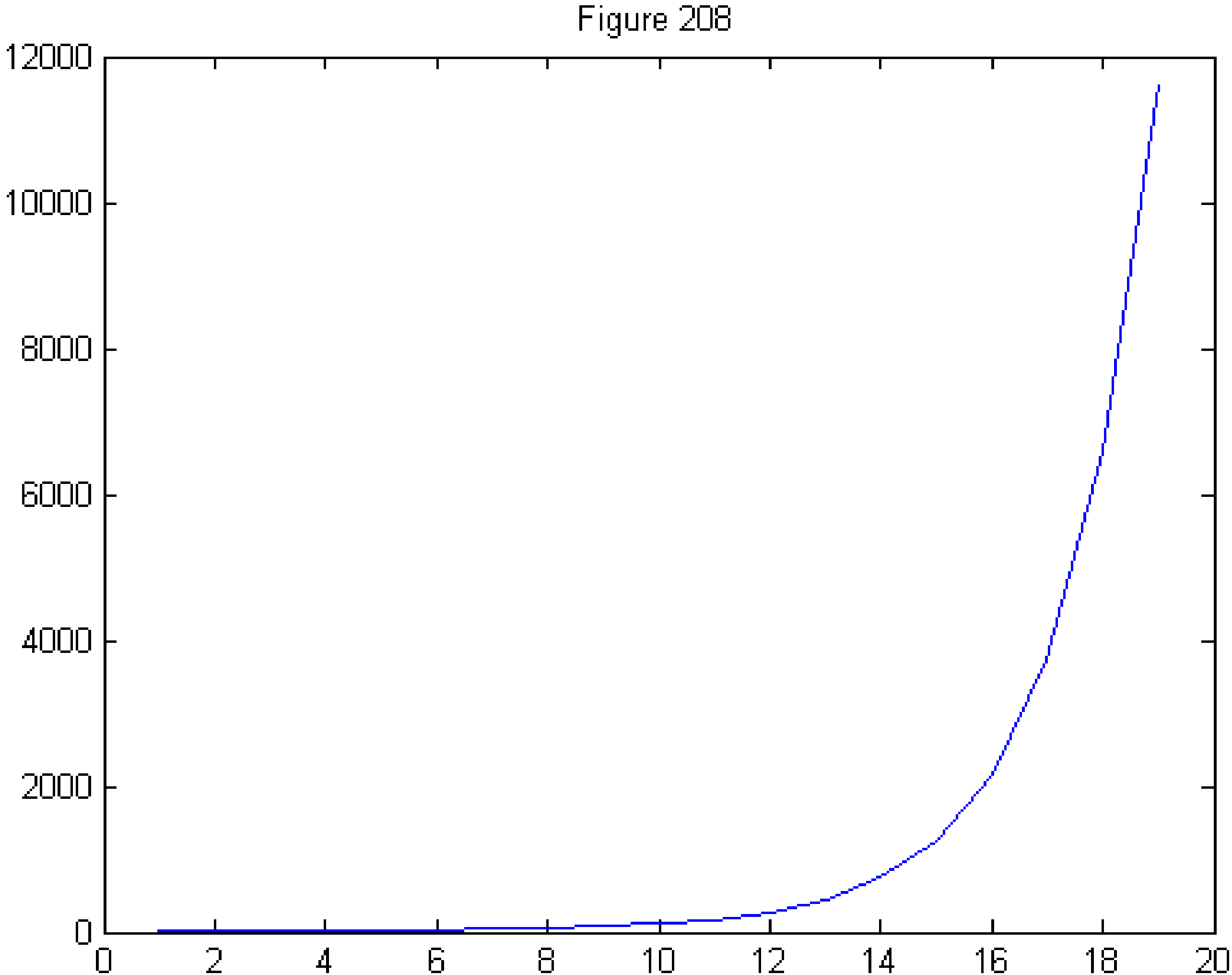


Figure 208
12000
10000
8000
6000
4000
2000
0
0
2
4
6
8
10
12
14
16
18
20

Figure 209

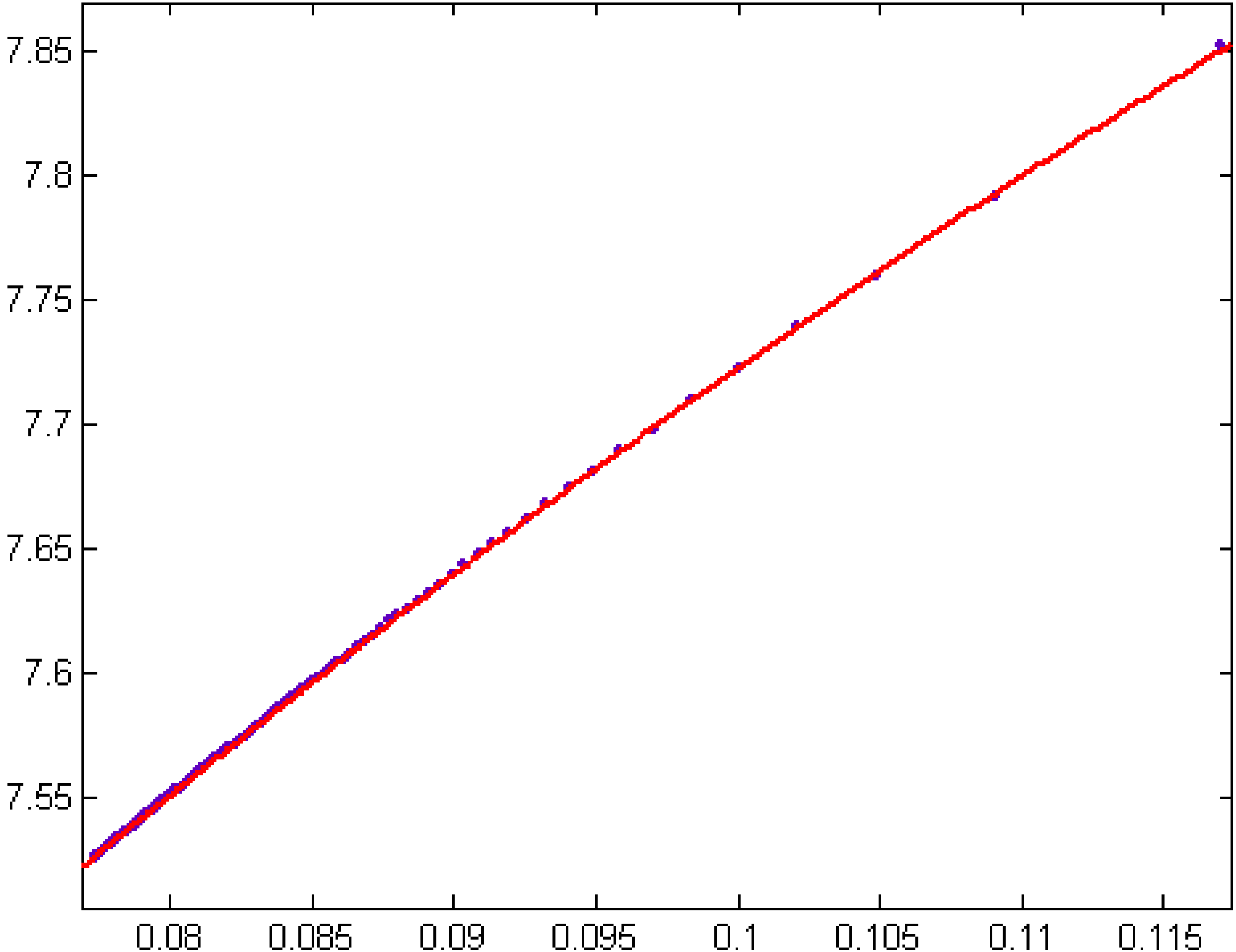


Figure 210

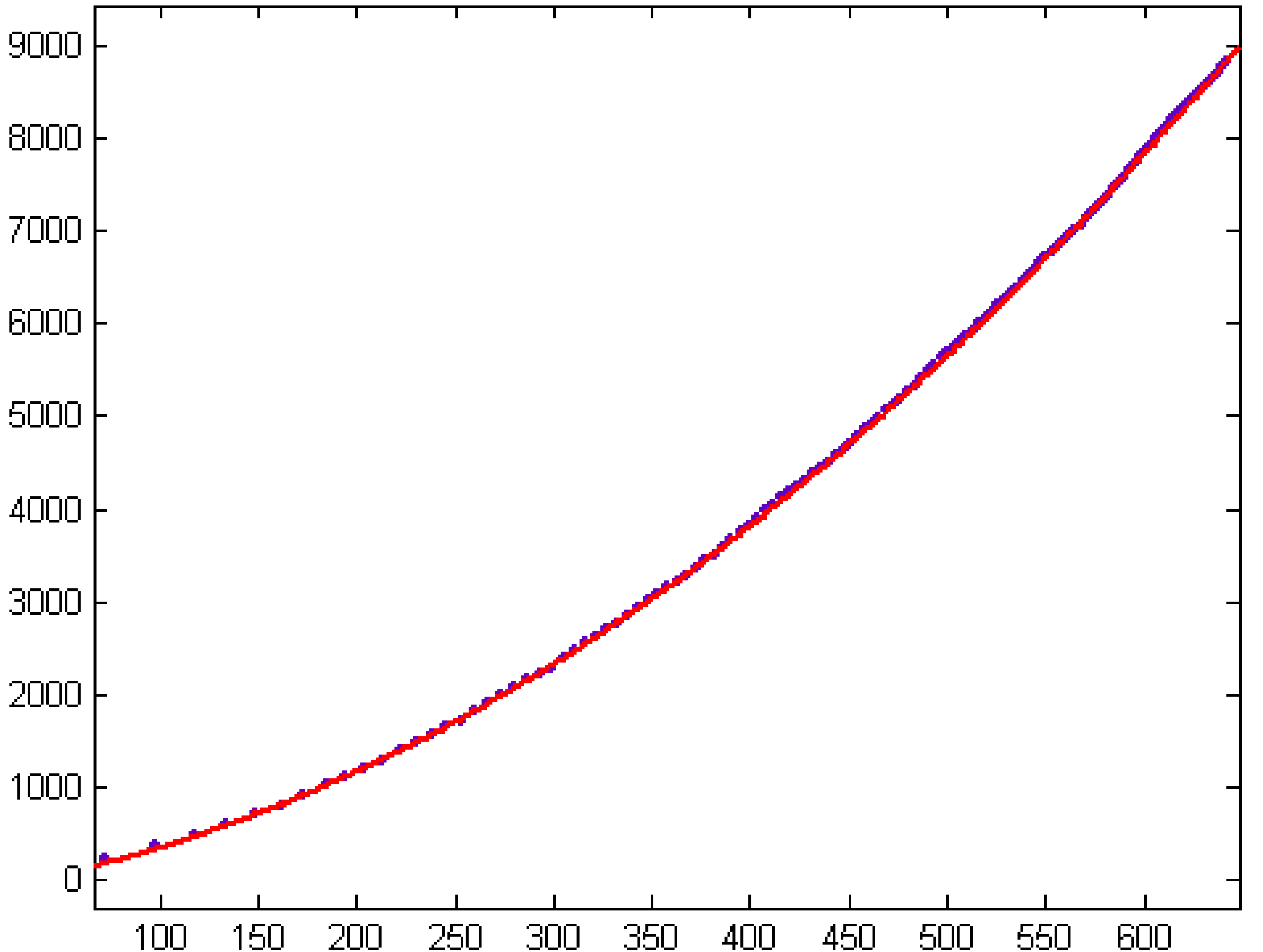

9000
8000
7000
6000
5000
4000
3000
2000
1000
0
100
150
200
250
300
350
400
450
500
550
600

Figure 211

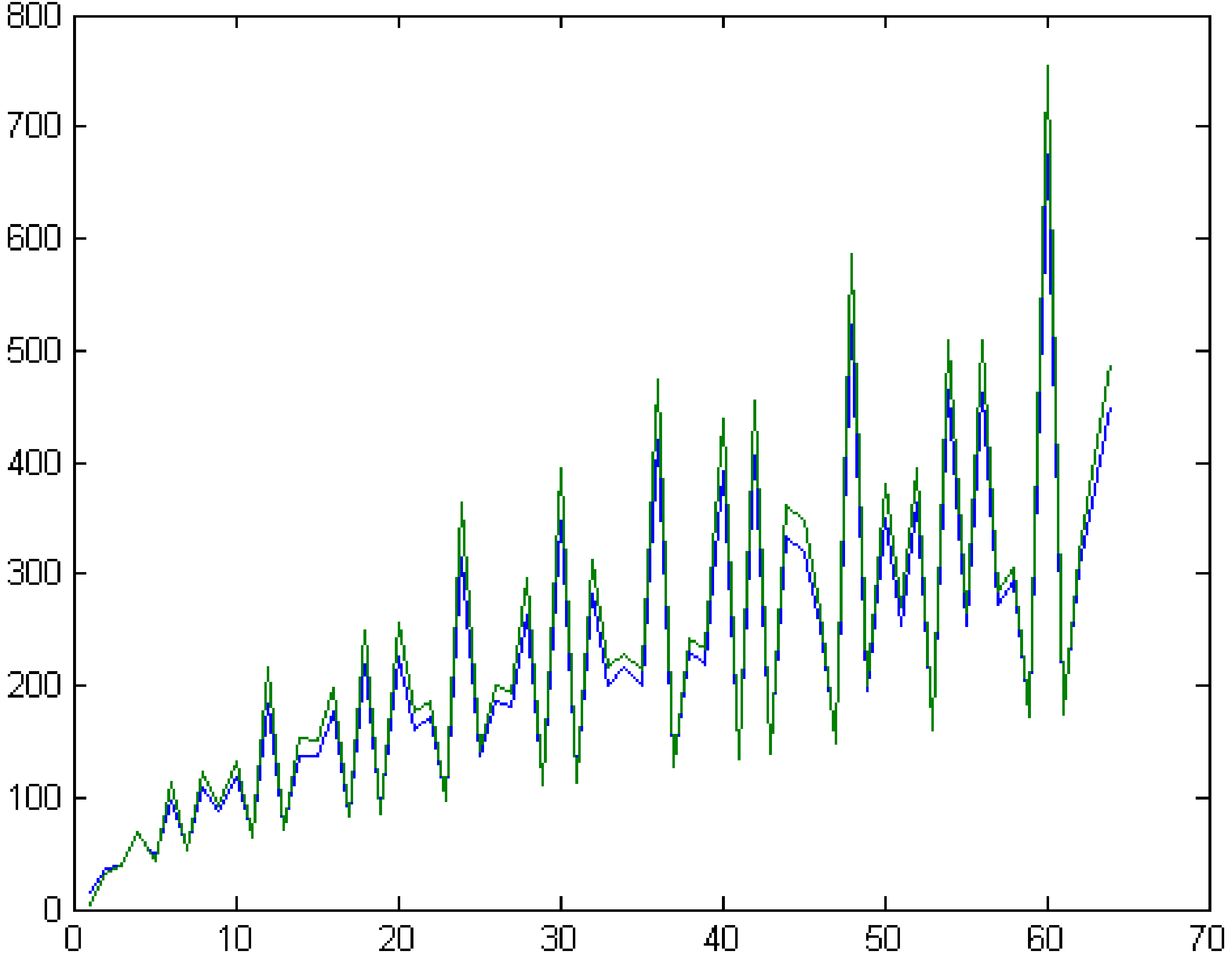
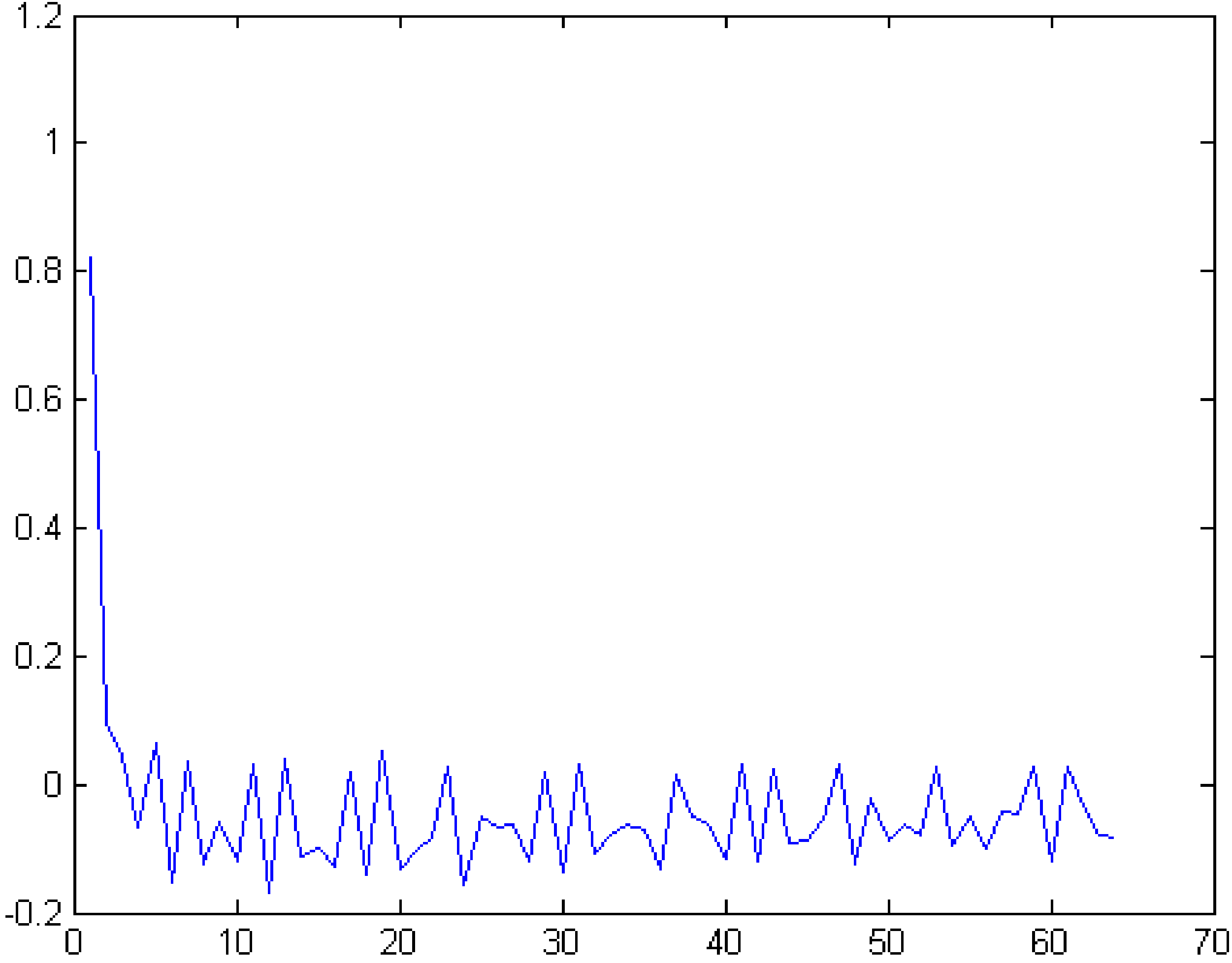


Figure 212



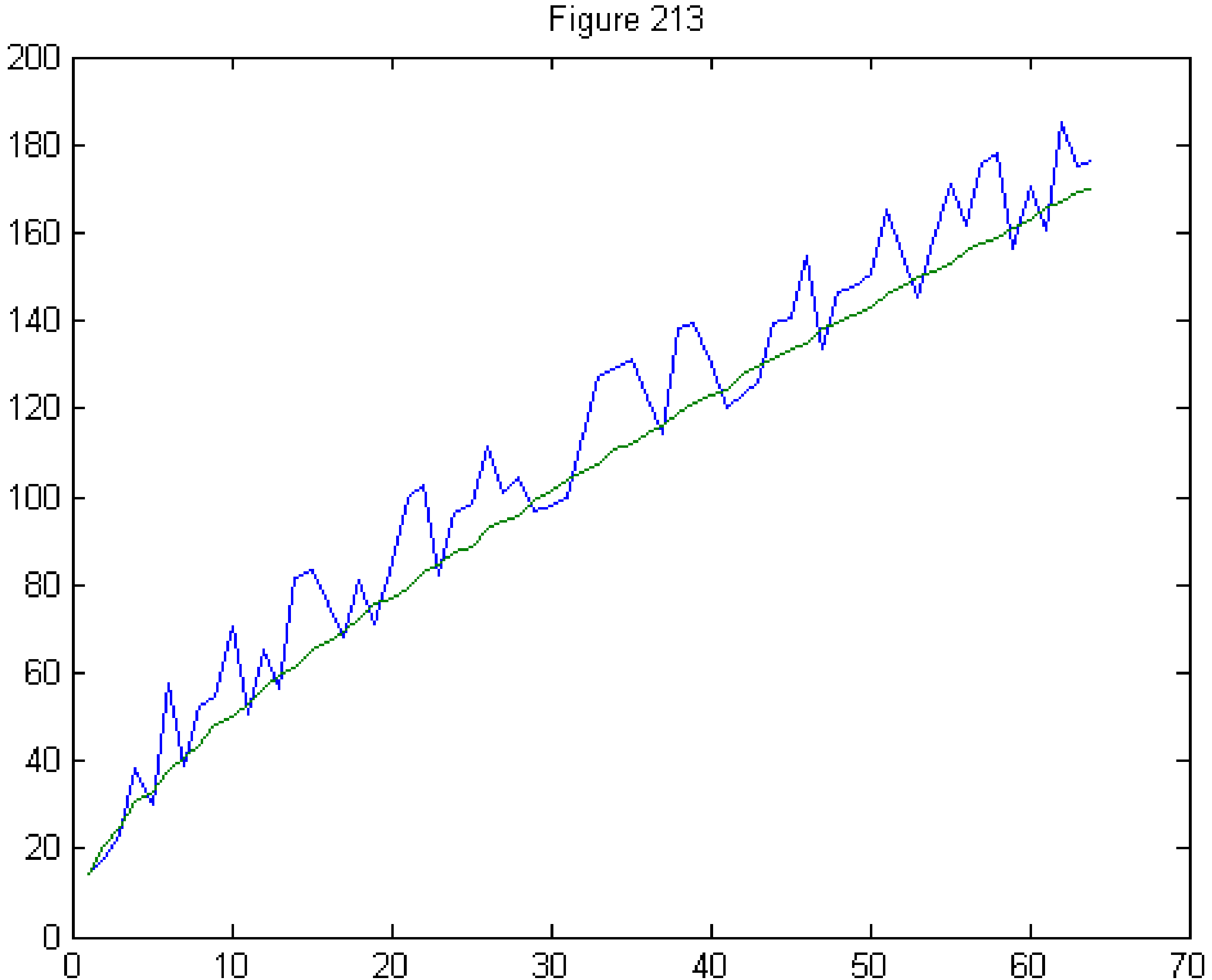


Figure 213
200
180
160
140
120
100
80
60
40
20
0
0
10
20
30
40
50
60
70

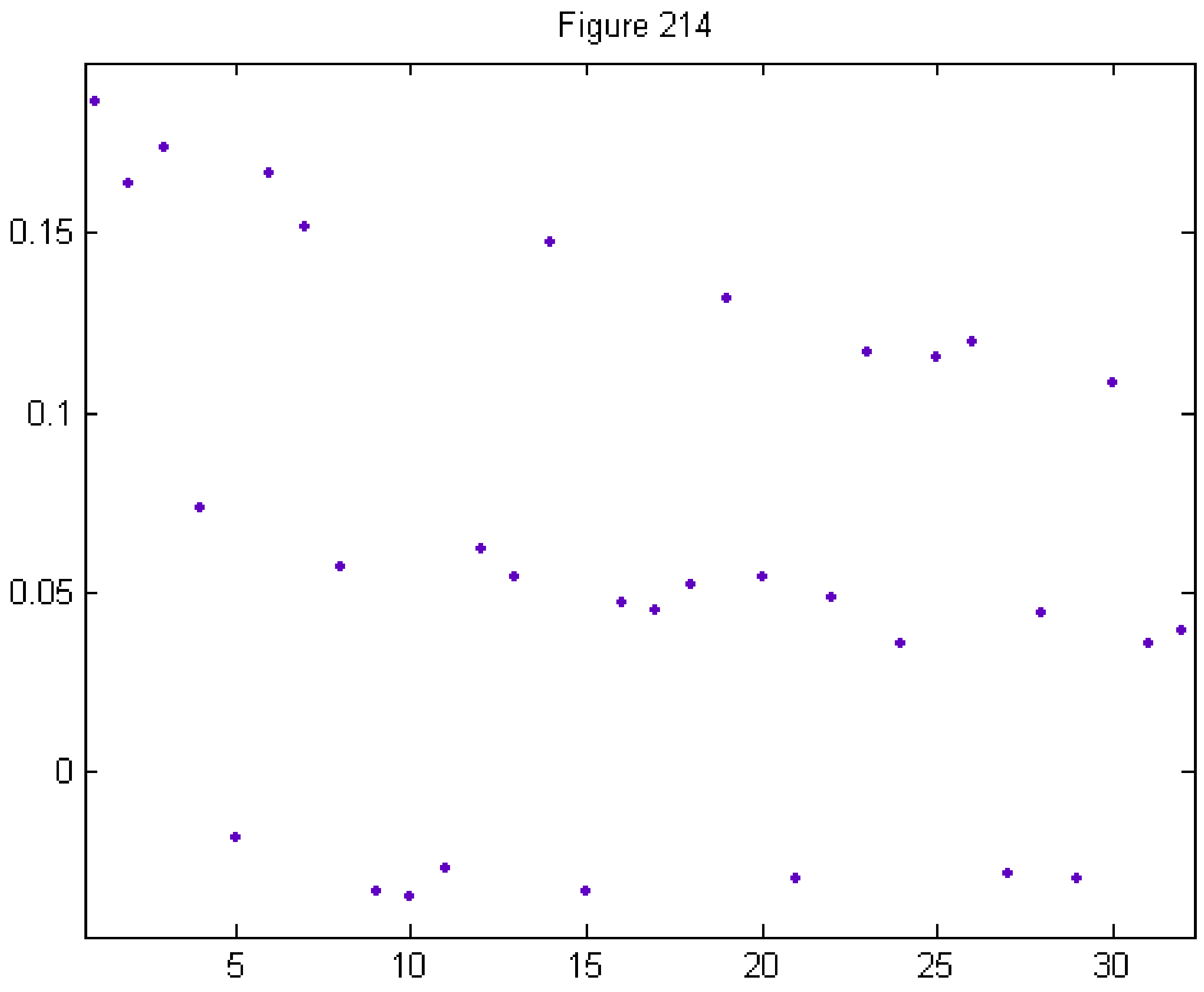


Figure 214
0.15
0.1
0.05
0
5
10
15
20
25
30

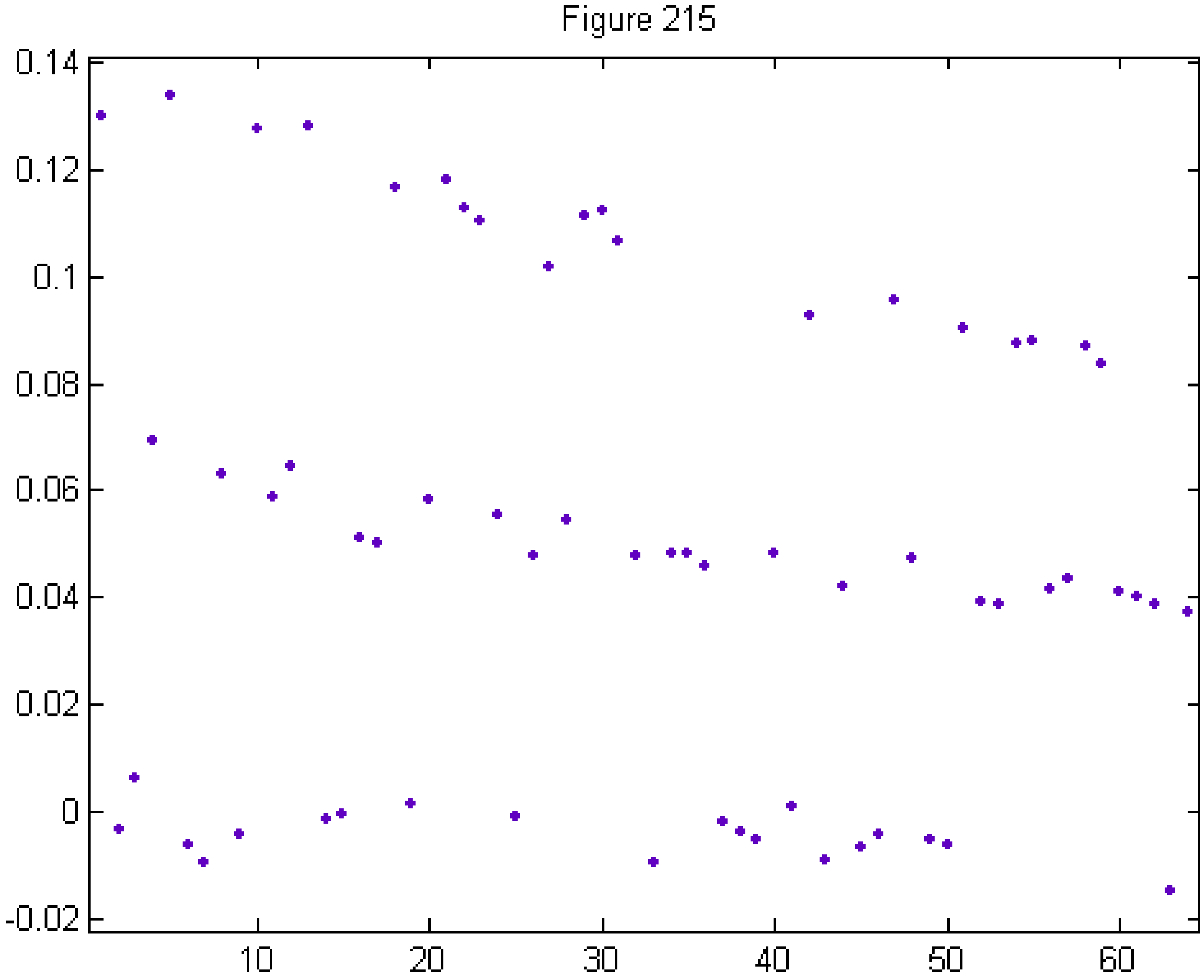
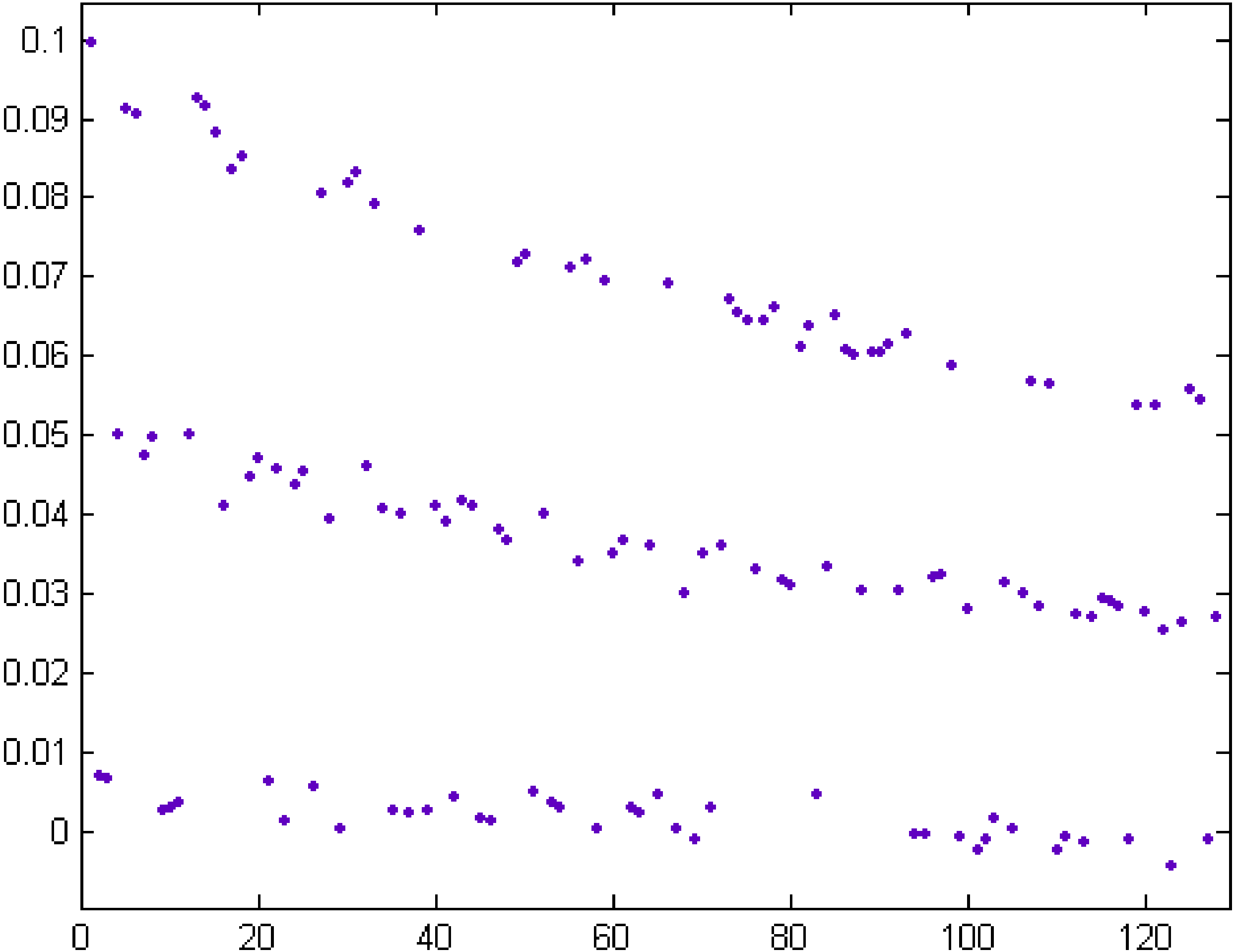


Figure 215
0.14
0.12
0.1
0.08
0.06
0.04
0.02
0
-0.02
10
20
30
40
50
60

Figure 216



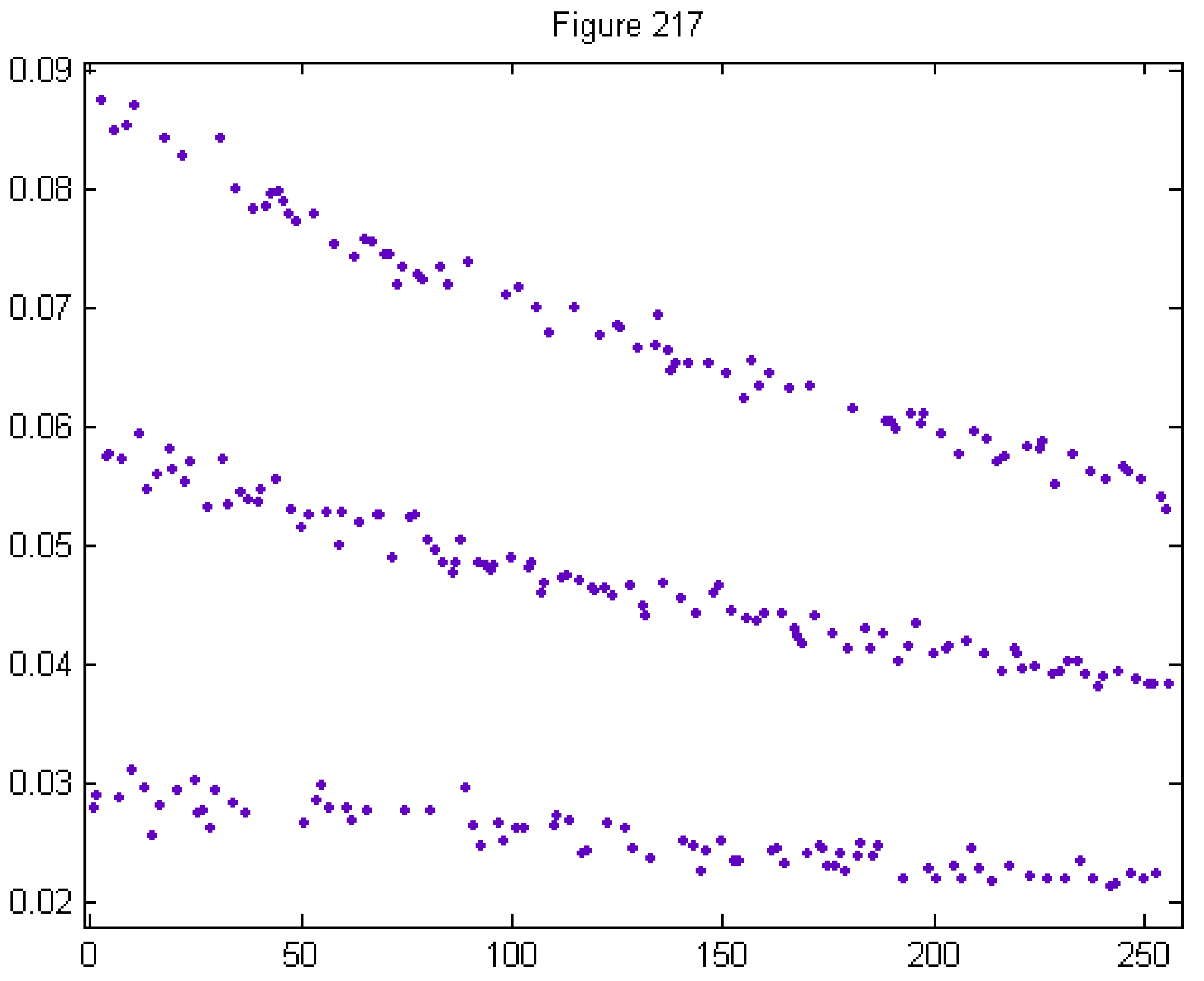
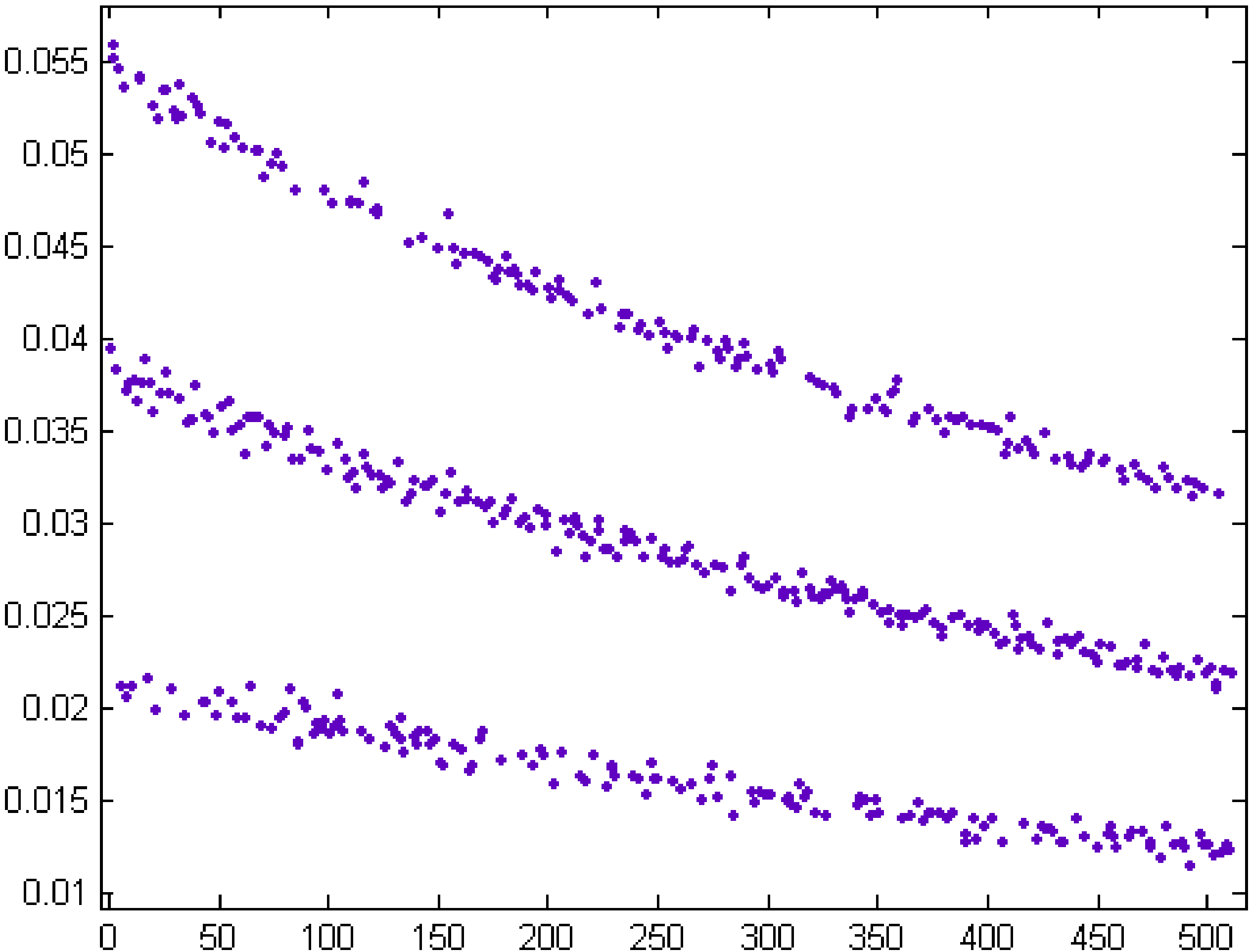


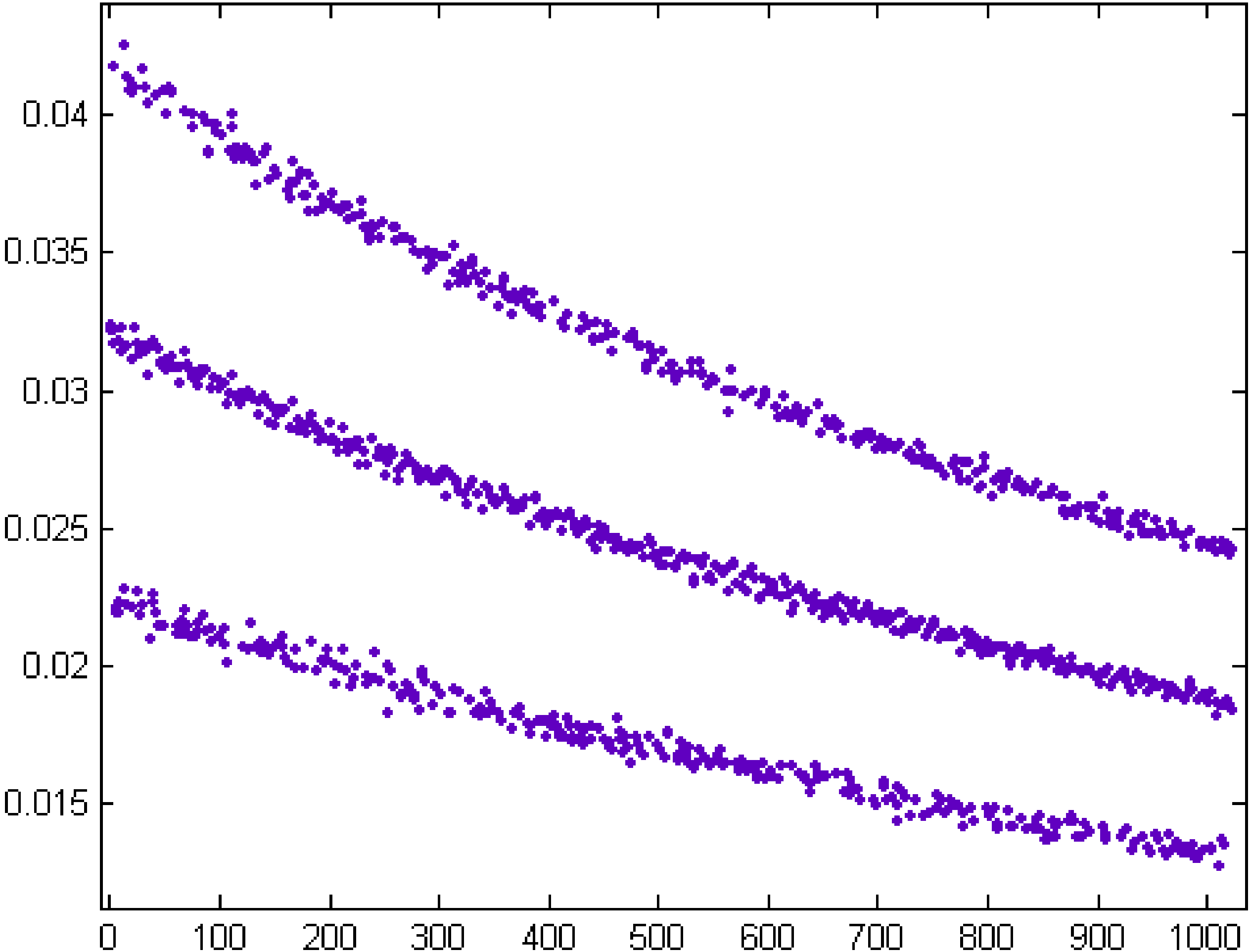
Figure 217
0.09
0.08
0.07
0.06
0.05
0.04
0.03
0.02
0
50
100
150
200
250

Figure 218



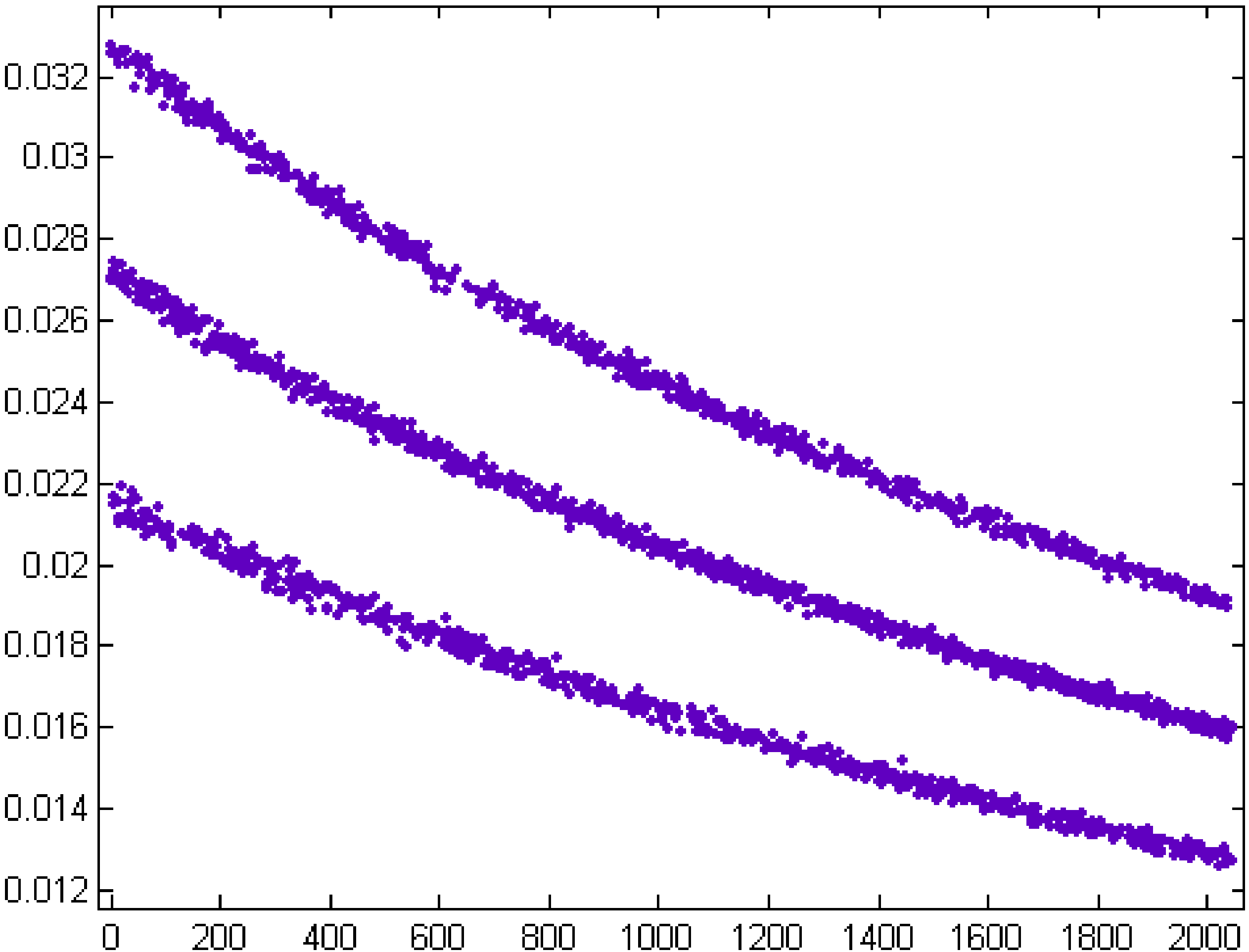
0.055
0.05
0.045
0.04
0.035
0.03
0.025
0.02
0.015
0.01
0
50
100
150
200
250
300
350
400
450
500

Figure 219



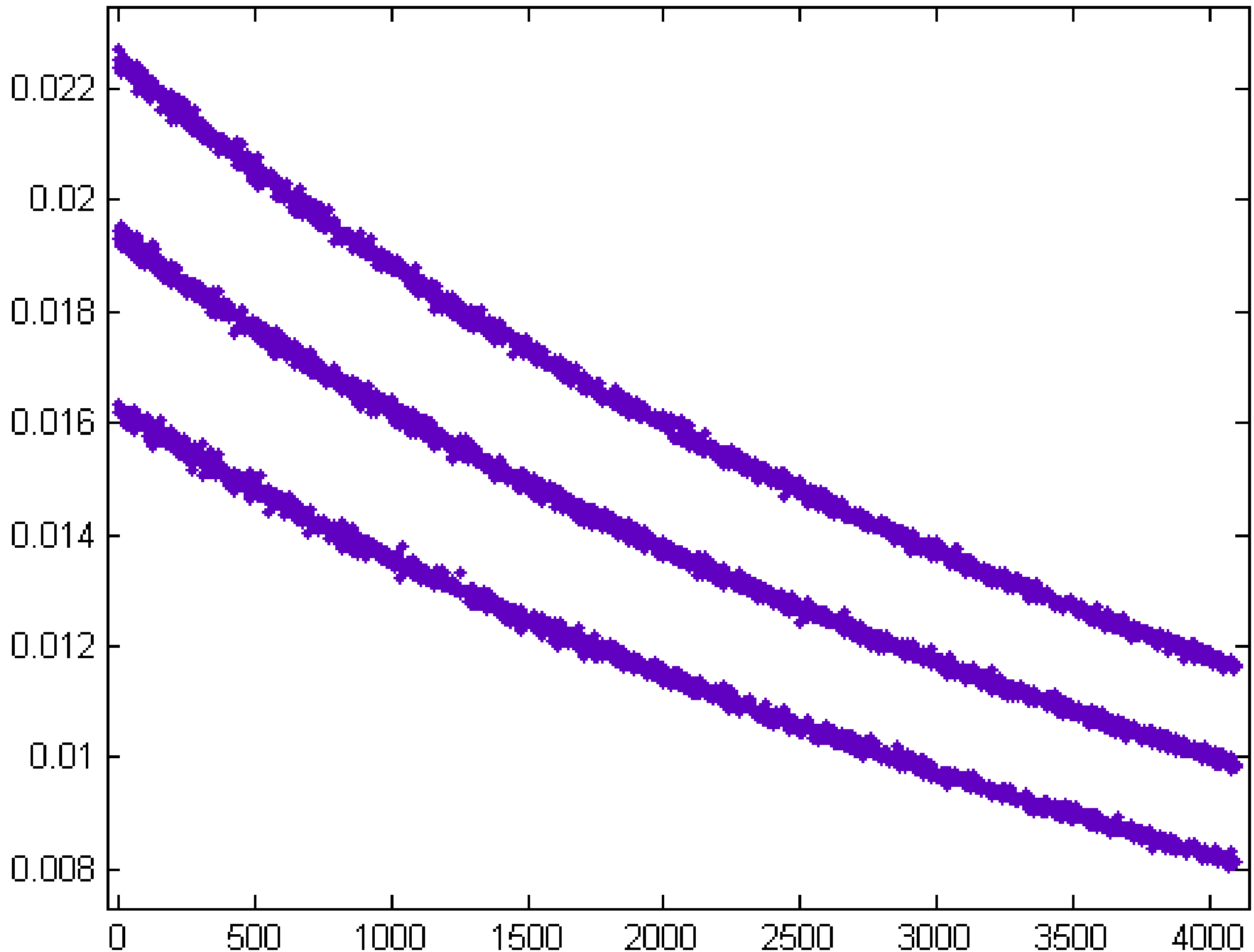
0.04
0.035
0.03
0.025
0.02
0.015
0
100
200
300
400
500
600
700
800
900
1000

Figure 220



0.032
0.03
0.028
0.026
0.024
0.022
0.02
0.018
0.016
0.014
0.012
0
200
400
600
800
1000
1200
1400
1600
1800
2000

Figure 221


0.022
0.02
0.018
0.016
0.014
0.012
0.01
0.008
0
500
1000
1500
2000
2500
3000
3500
4000

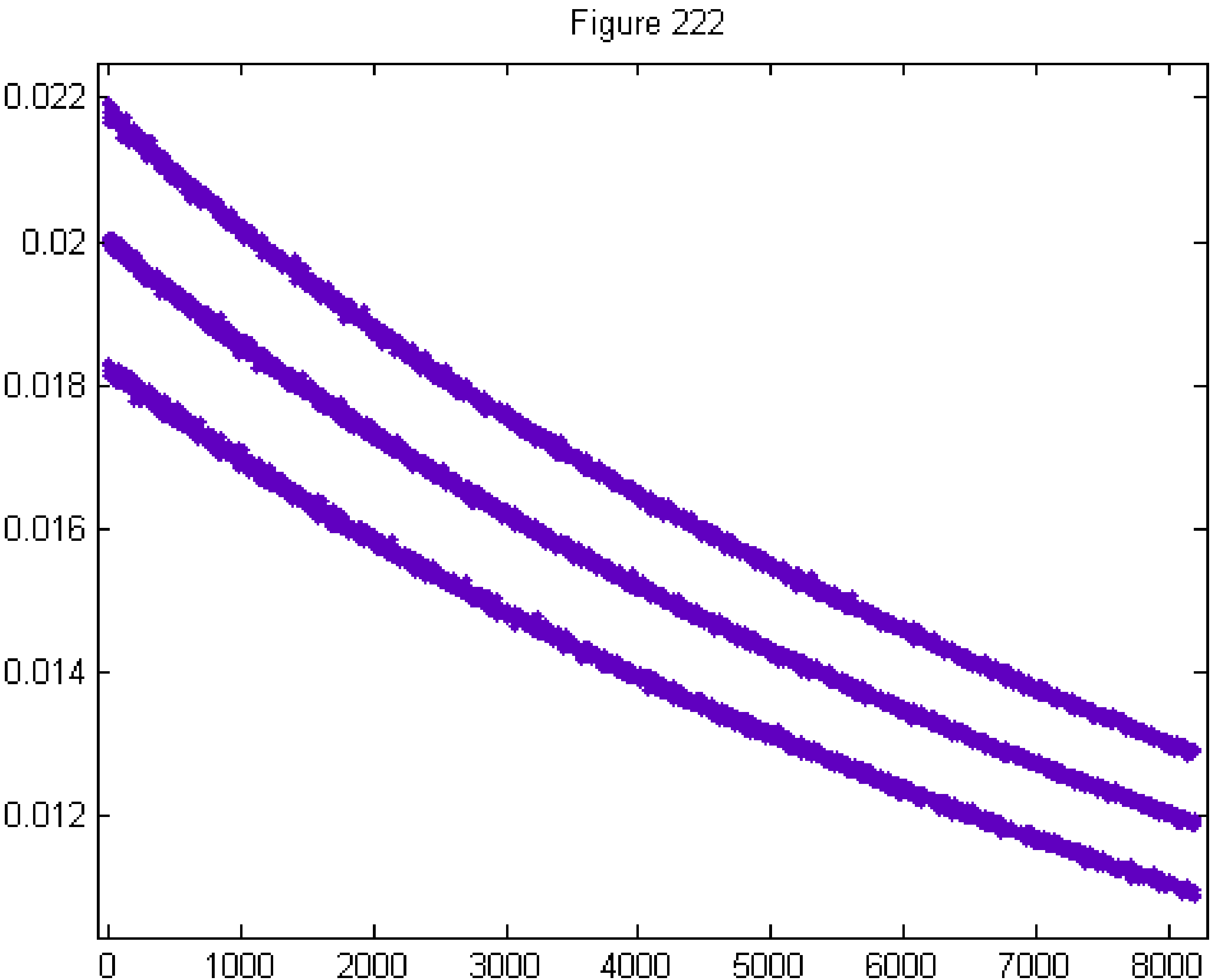


Figure 222
0.022
0.02
0.018
0.016
0.014
0.012
0
1000
2000
3000
4000
5000
6000
7000
8000

Figure 223

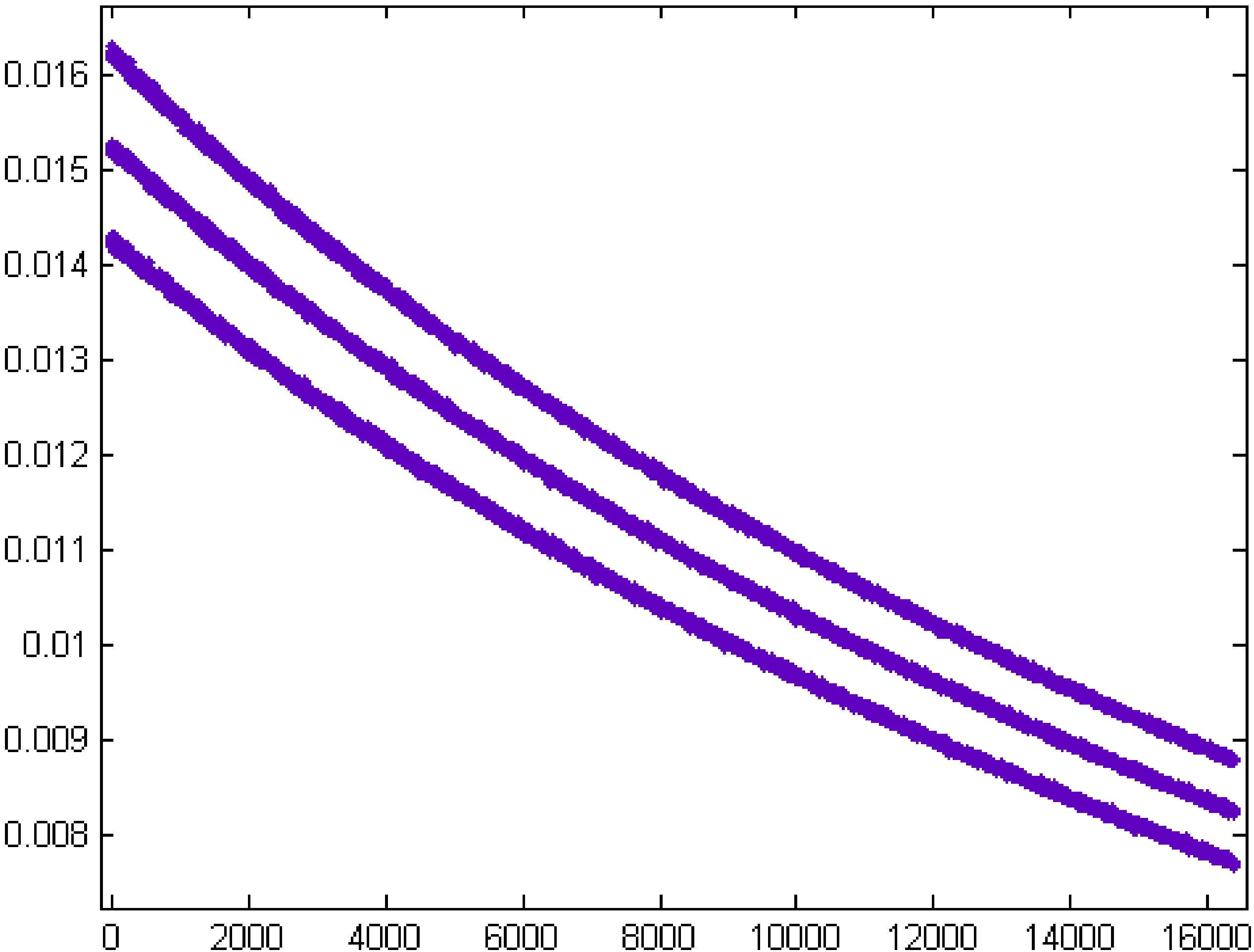

0.016
0.015
0.014
0.013
0.012
0.011
0.01
0.009
0.008
0
2000
4000
6000
8000
10000
12000
14000
16000

Figure 224

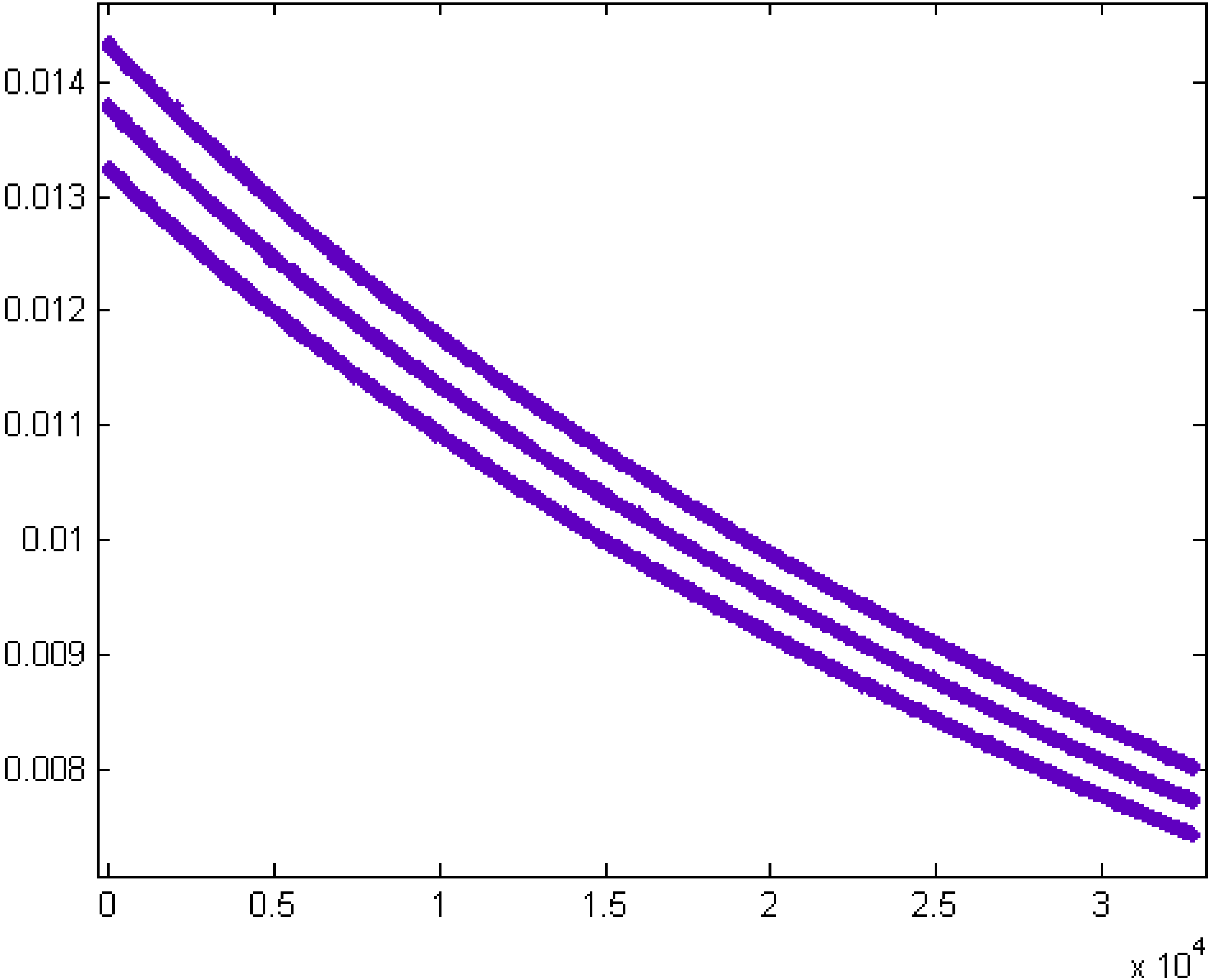
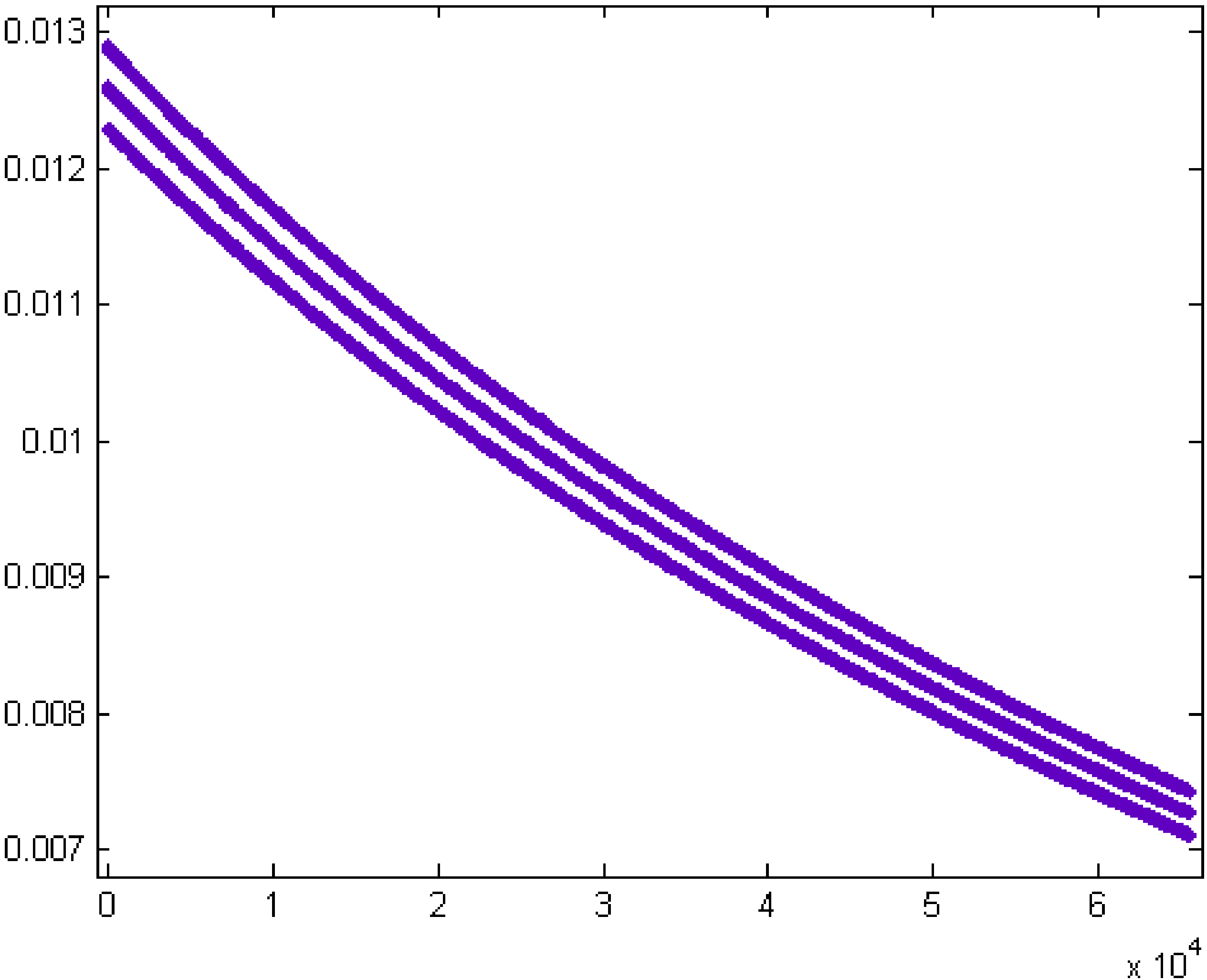


Figure 225



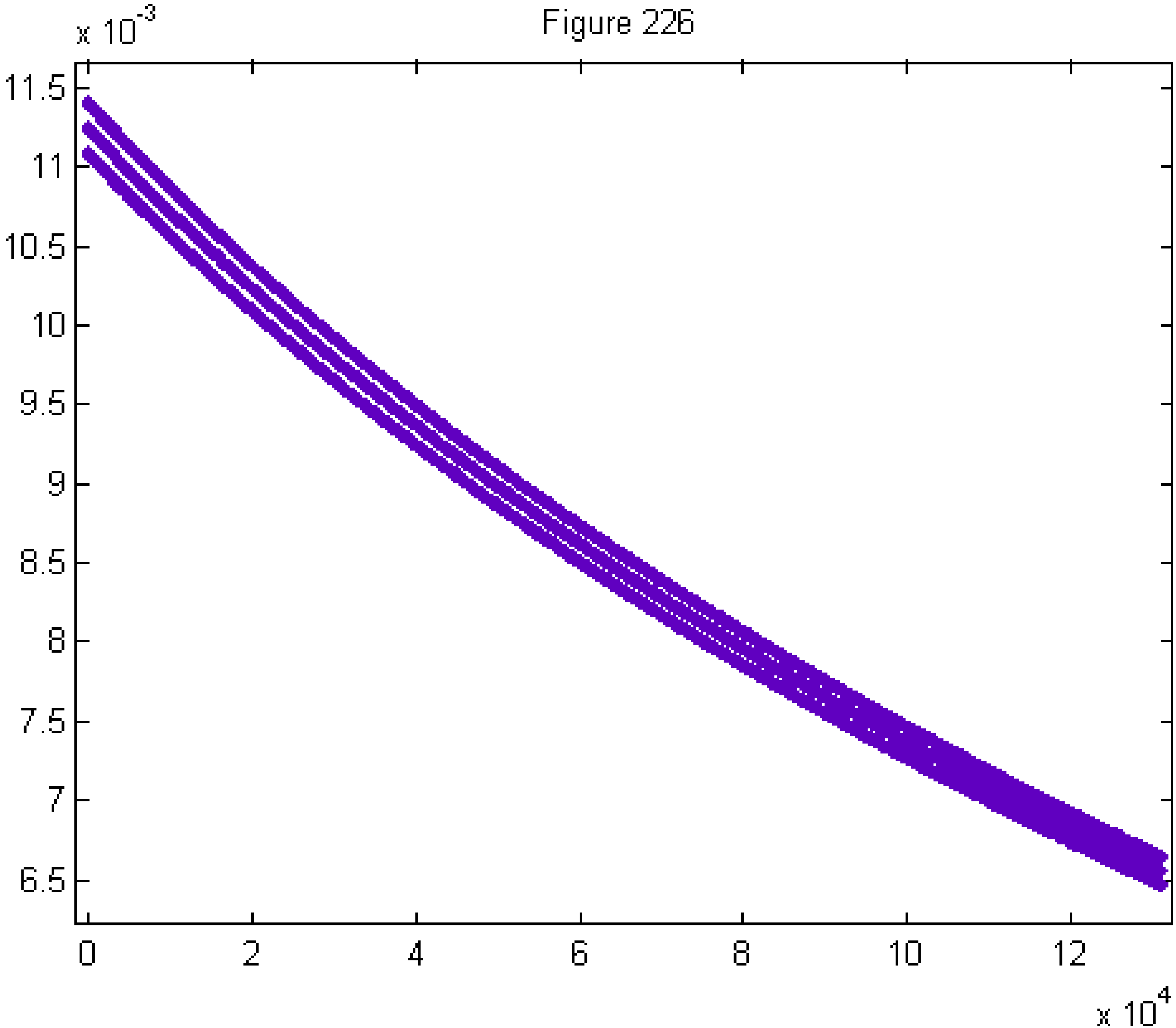


Figure 226
x 10^-3
11.5
11
10.5
10
9.5
9
8.5
8
7.5
7
6.5
0
2
4
6
8
10
12
x 10^4

Figure 227

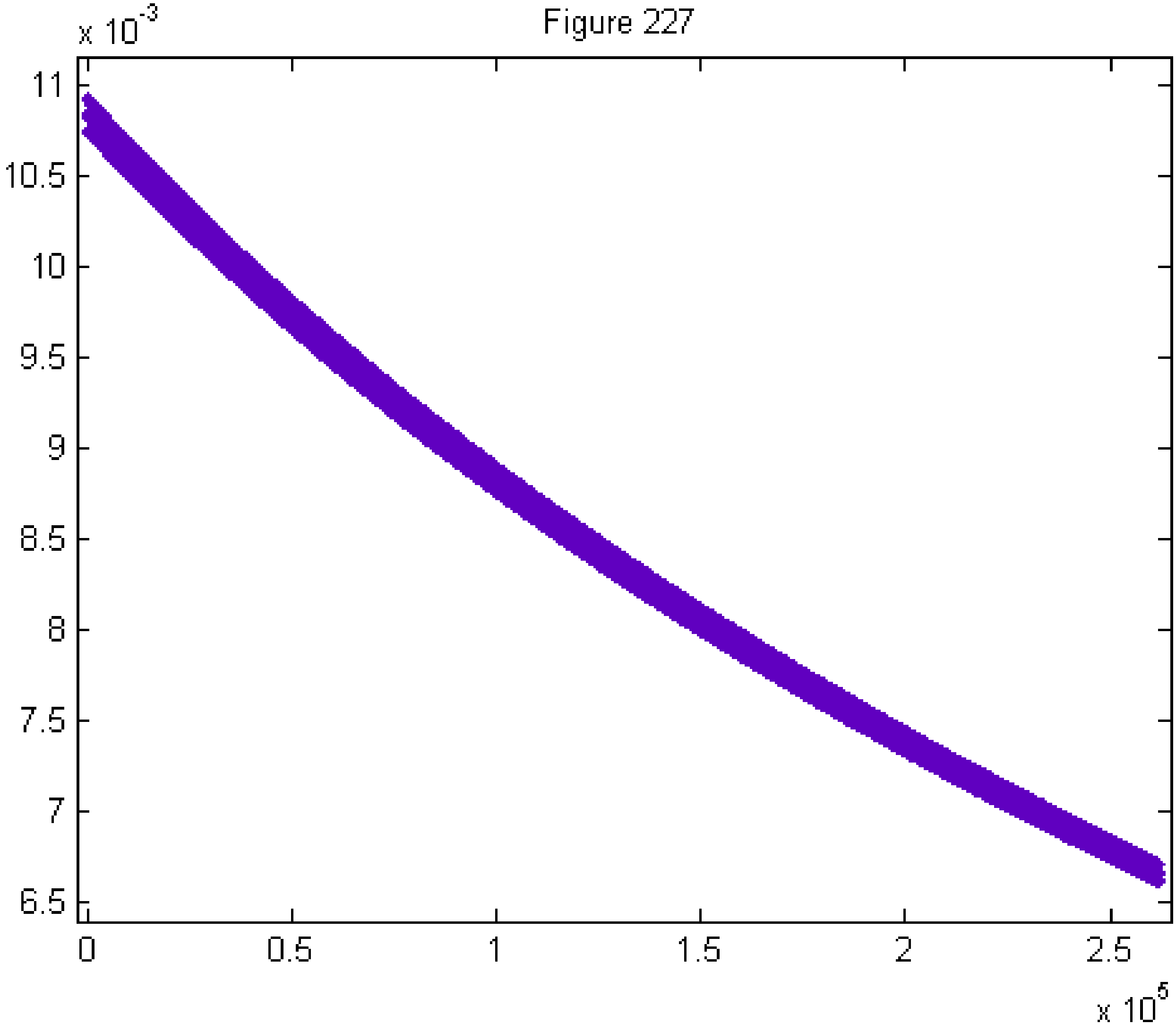
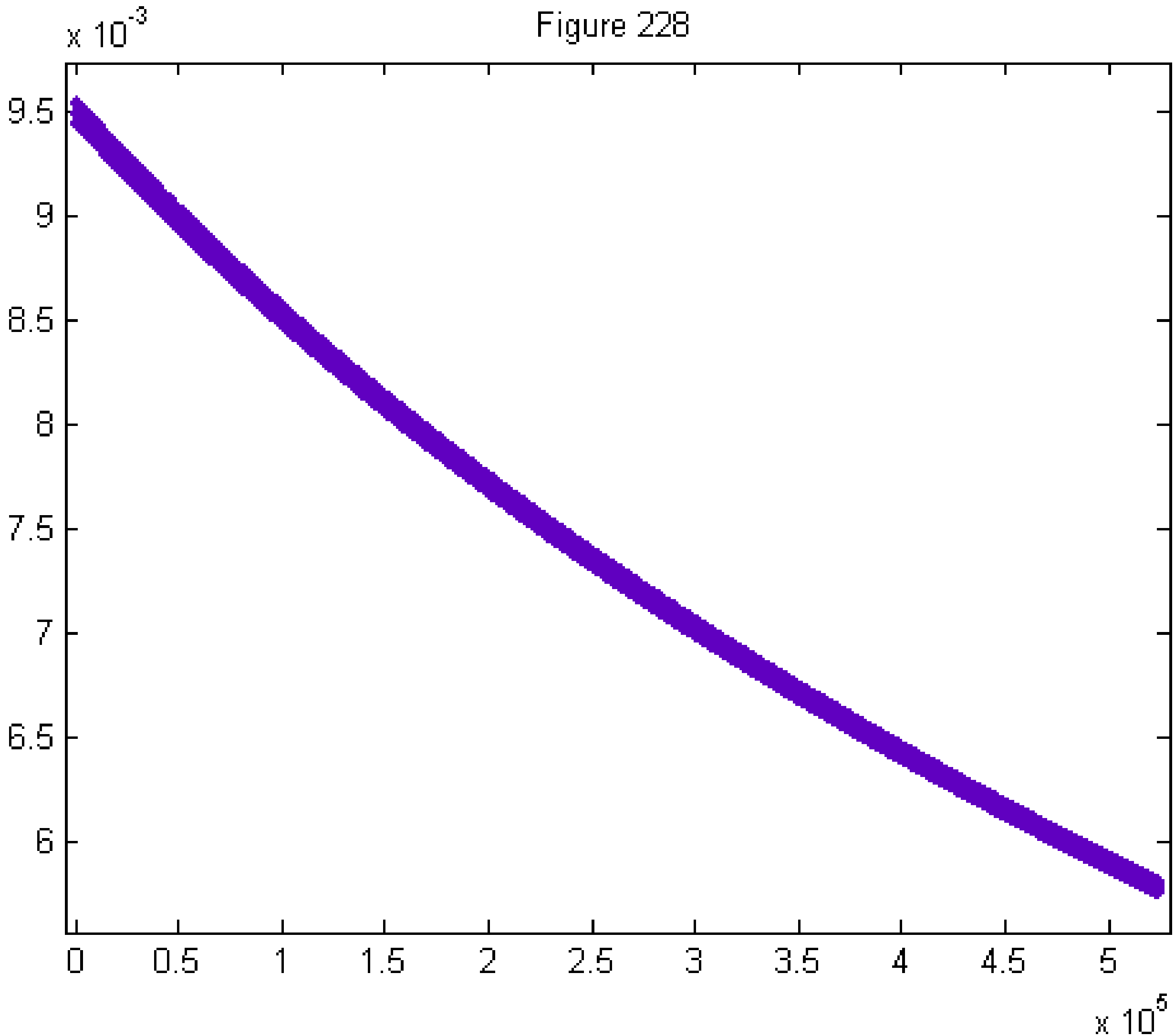


Figure 228



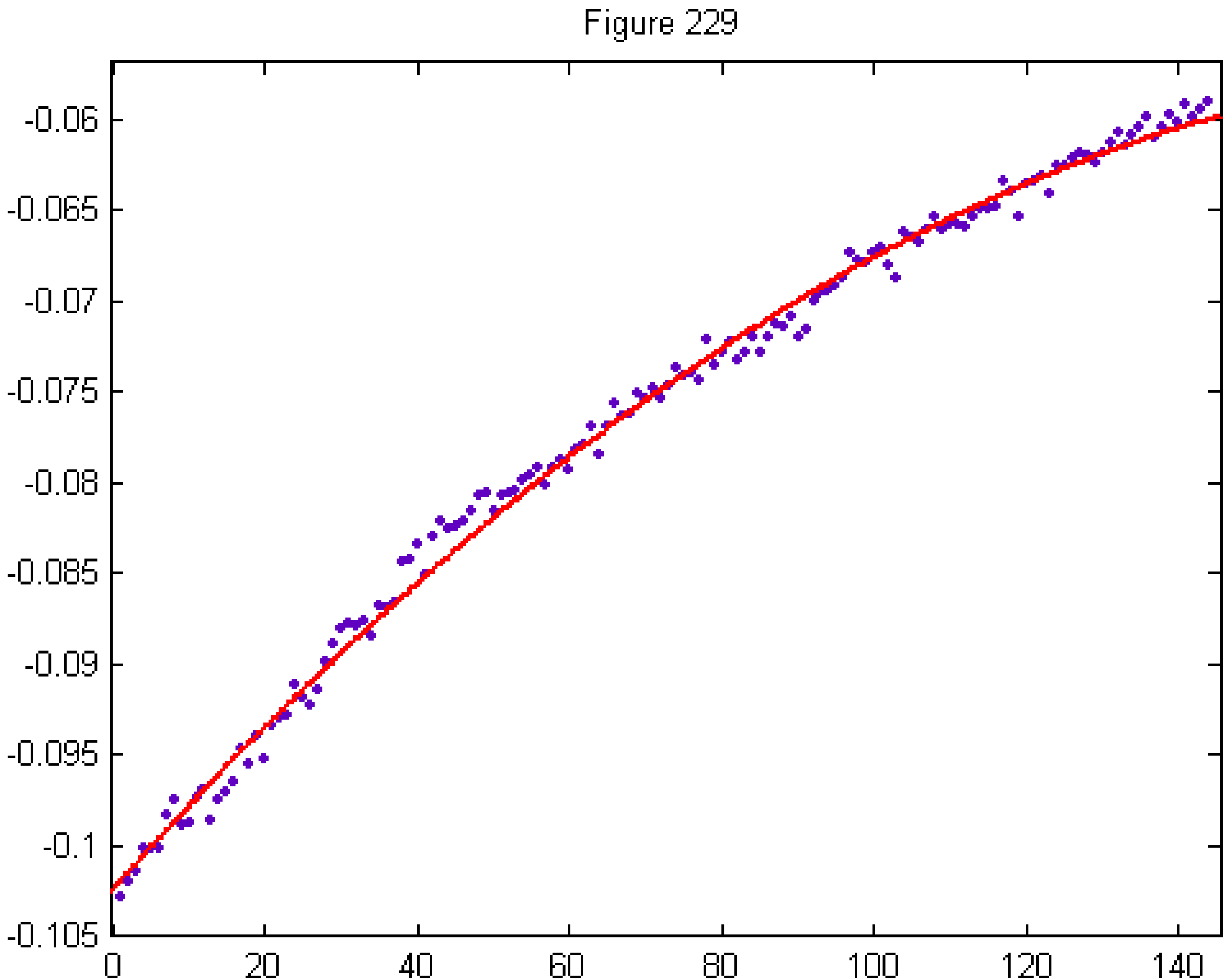
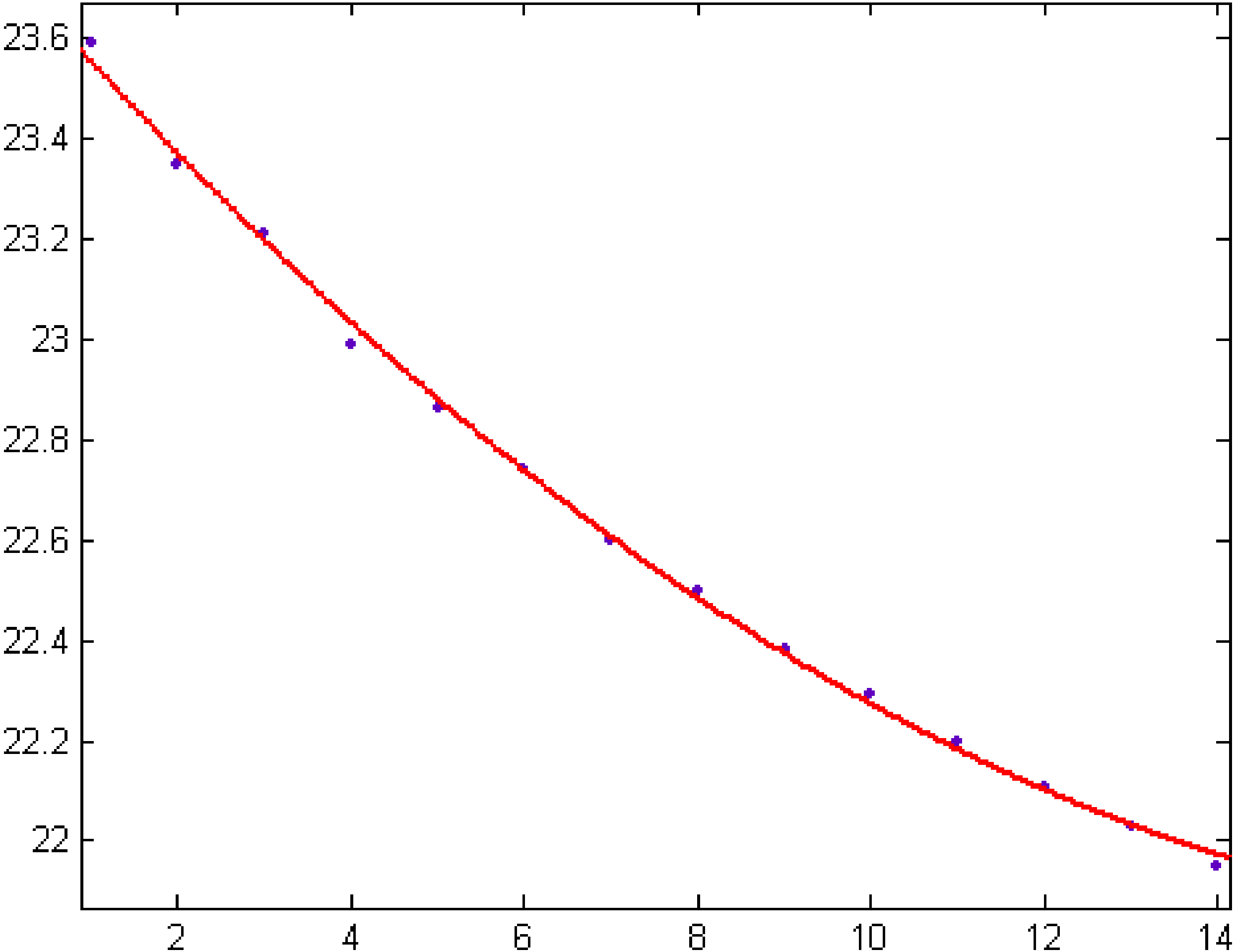


Figure 229
-0.06
-0.065
-0.07
-0.075
-0.08
-0.085
-0.09
-0.095
-0.1
-0.105
0
20
40
60
80
100
120
140

Figure 230



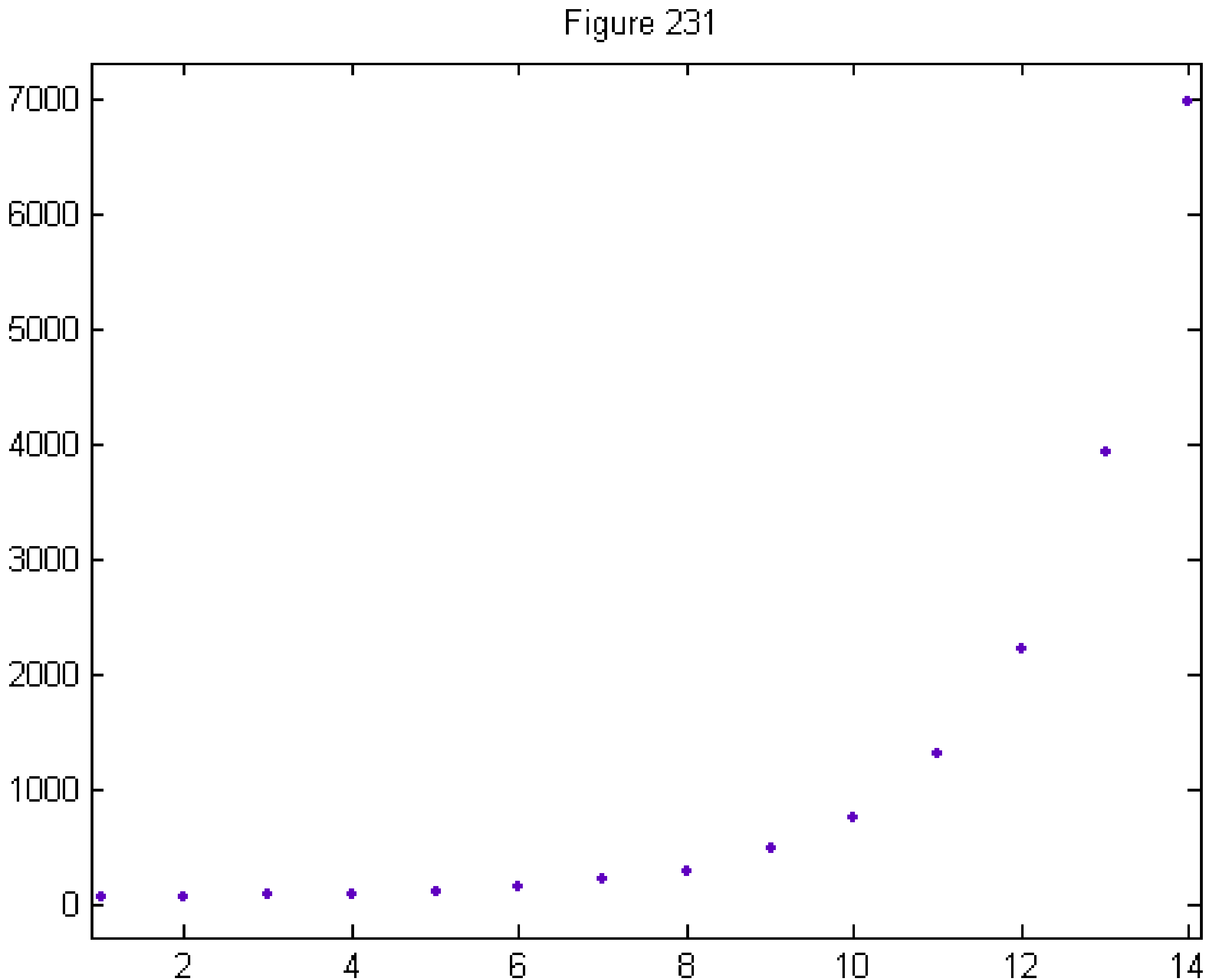


Figure 231
7000
6000
5000
4000
3000
2000
1000
0
2
4
6
8
10
12
14

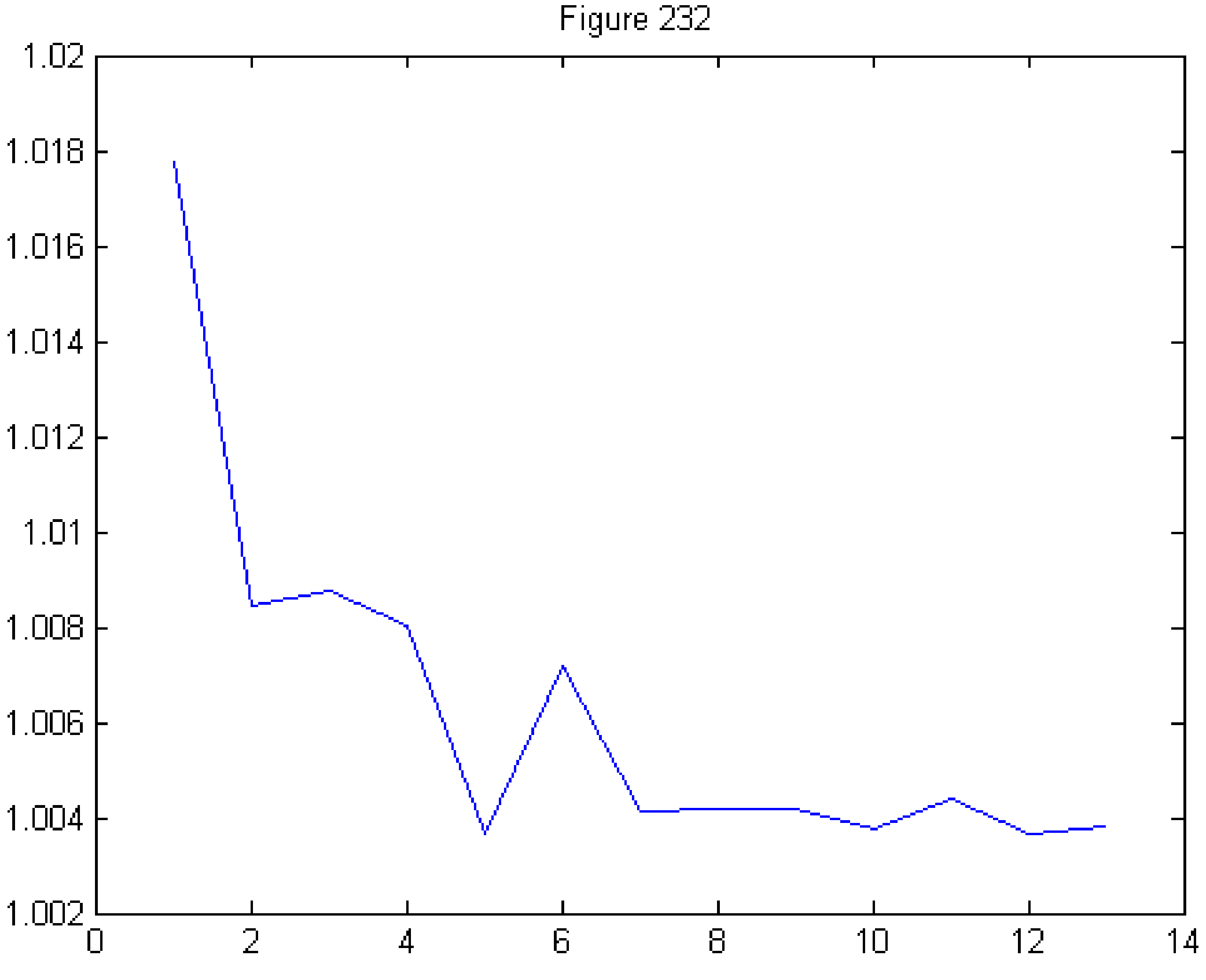
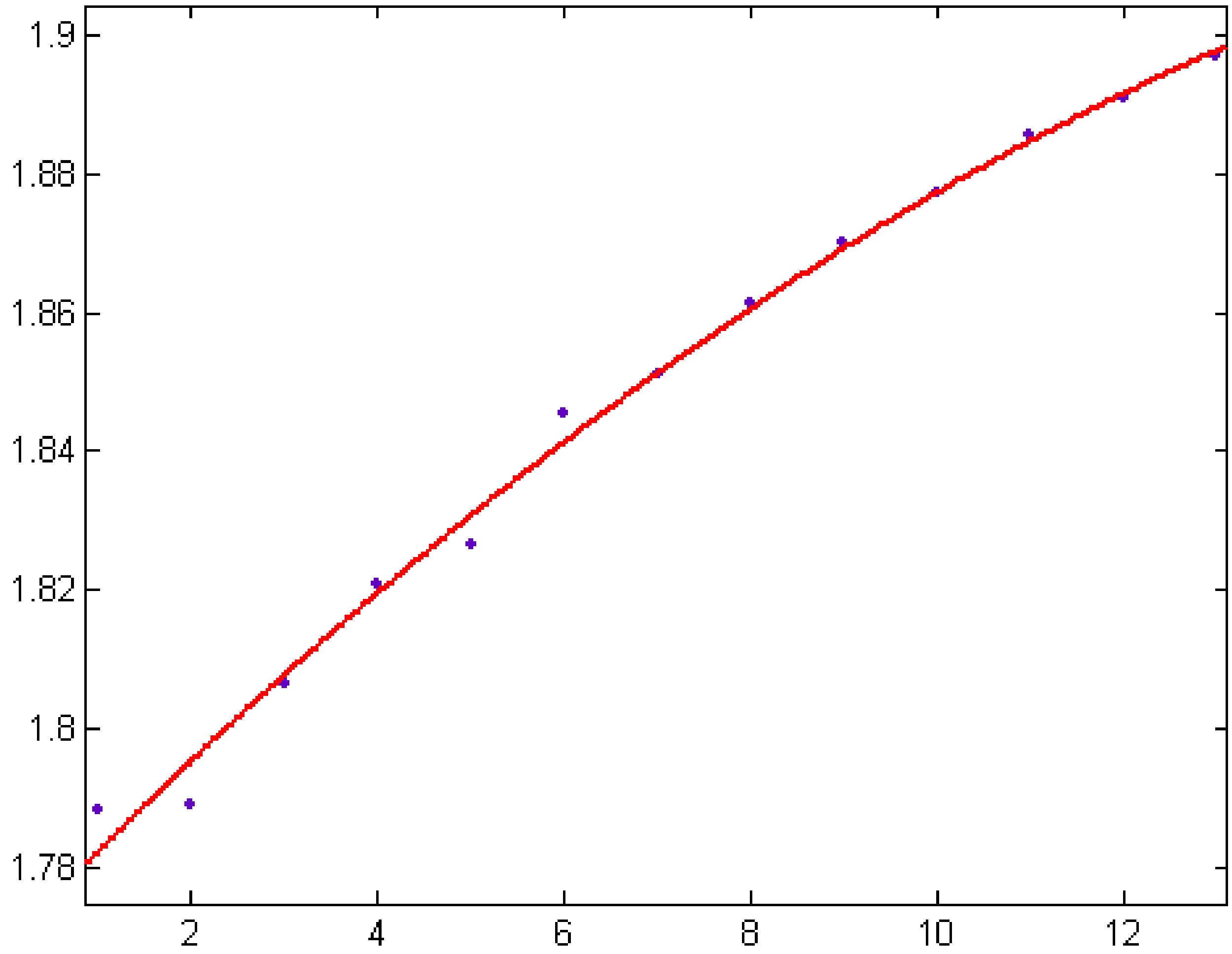


Figure 232
1.02
1.018
1.016
1.014
1.012
1.01
1.008
1.006
1.004
1.002
0
2
4
6
8
10
12
14

Figure 233



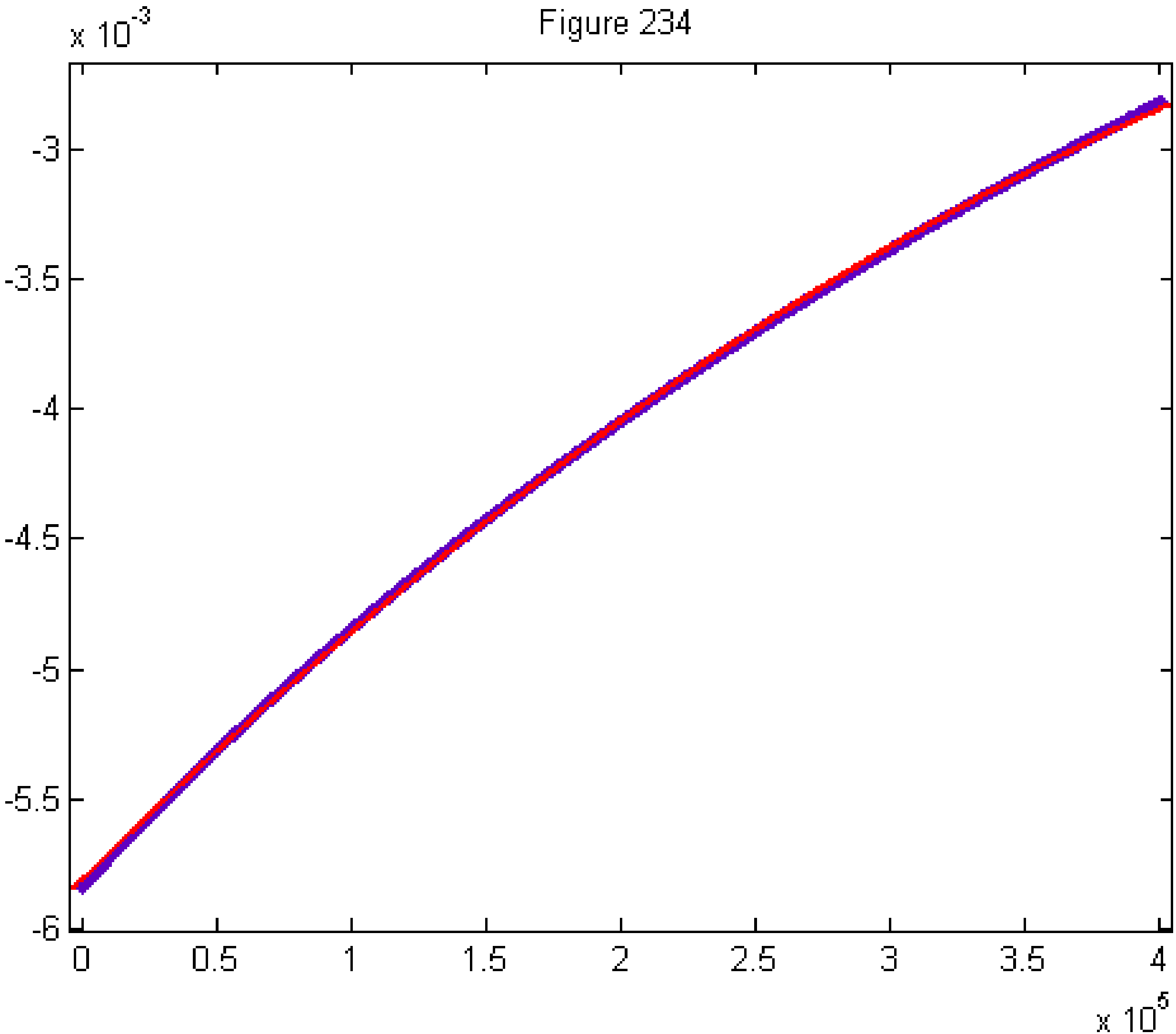


Figure 234
x 10^-3
-3
-3.5
-4
-4.5
-5
-5.5
-6
0
0.5
1
1.5
2
2.5
3
3.5
4
x 10^5

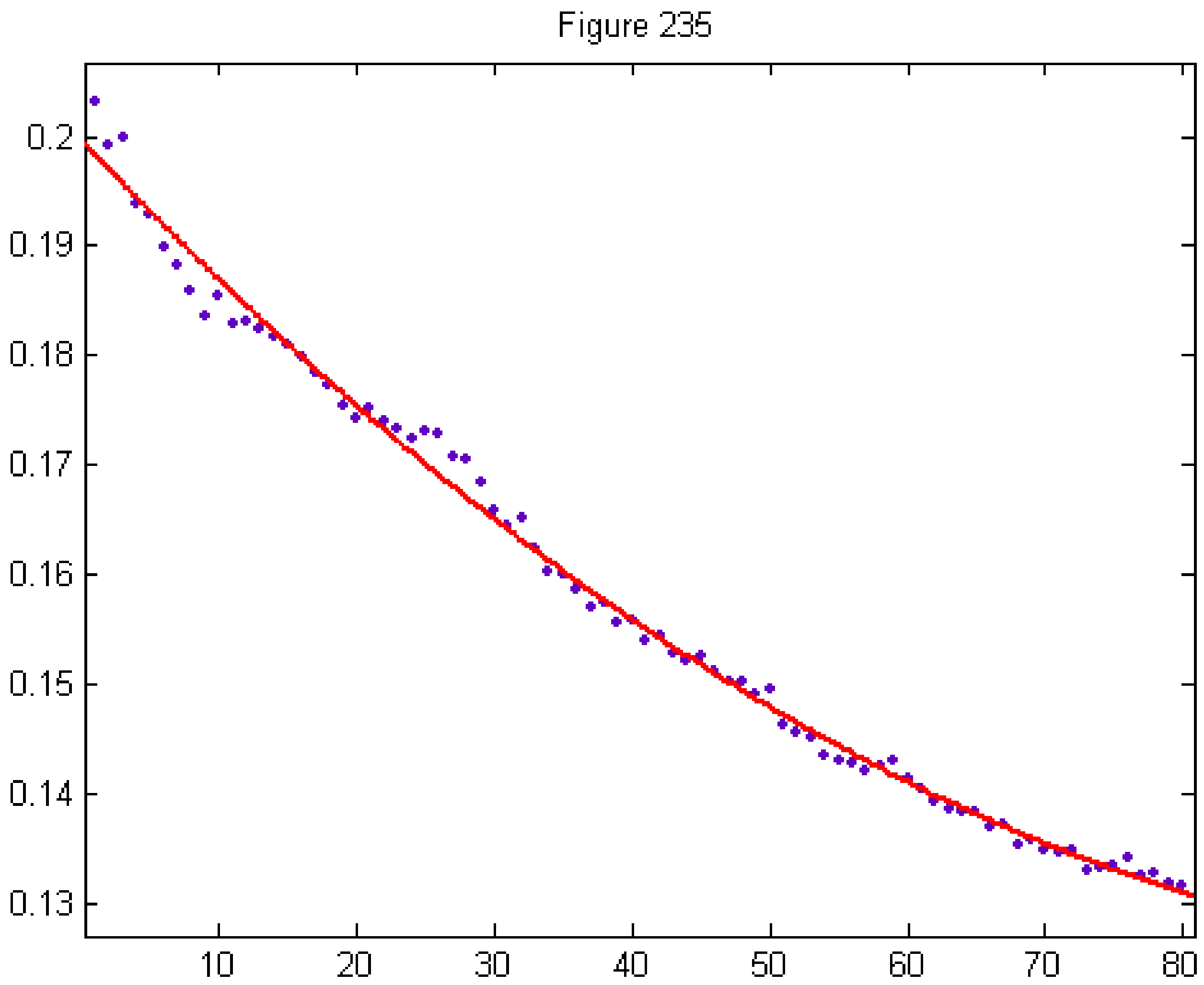


Figure 235
0.2
0.19
0.18
0.17
0.16
0.15
0.14
0.13
10
20
30
40
50
60
70
80

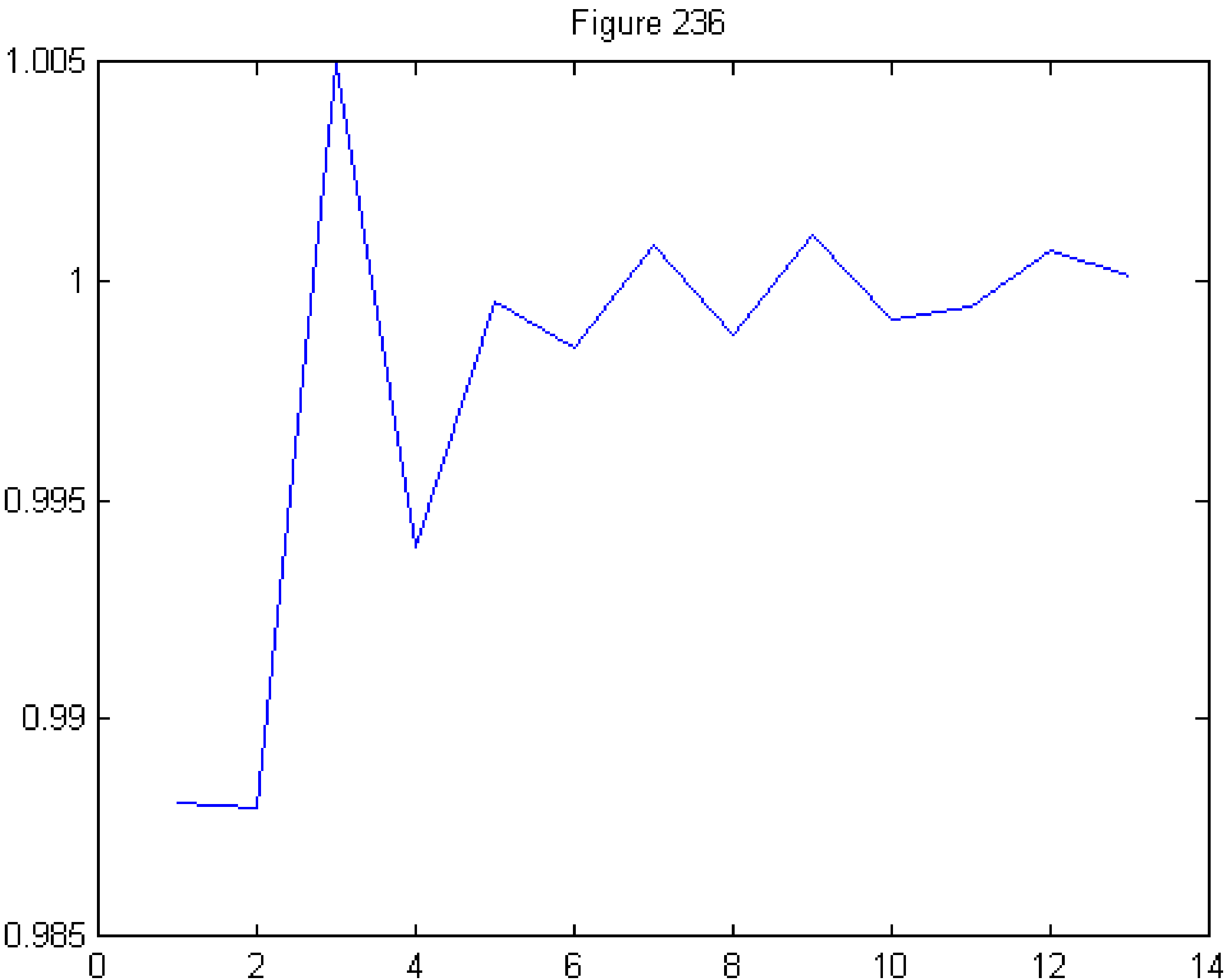
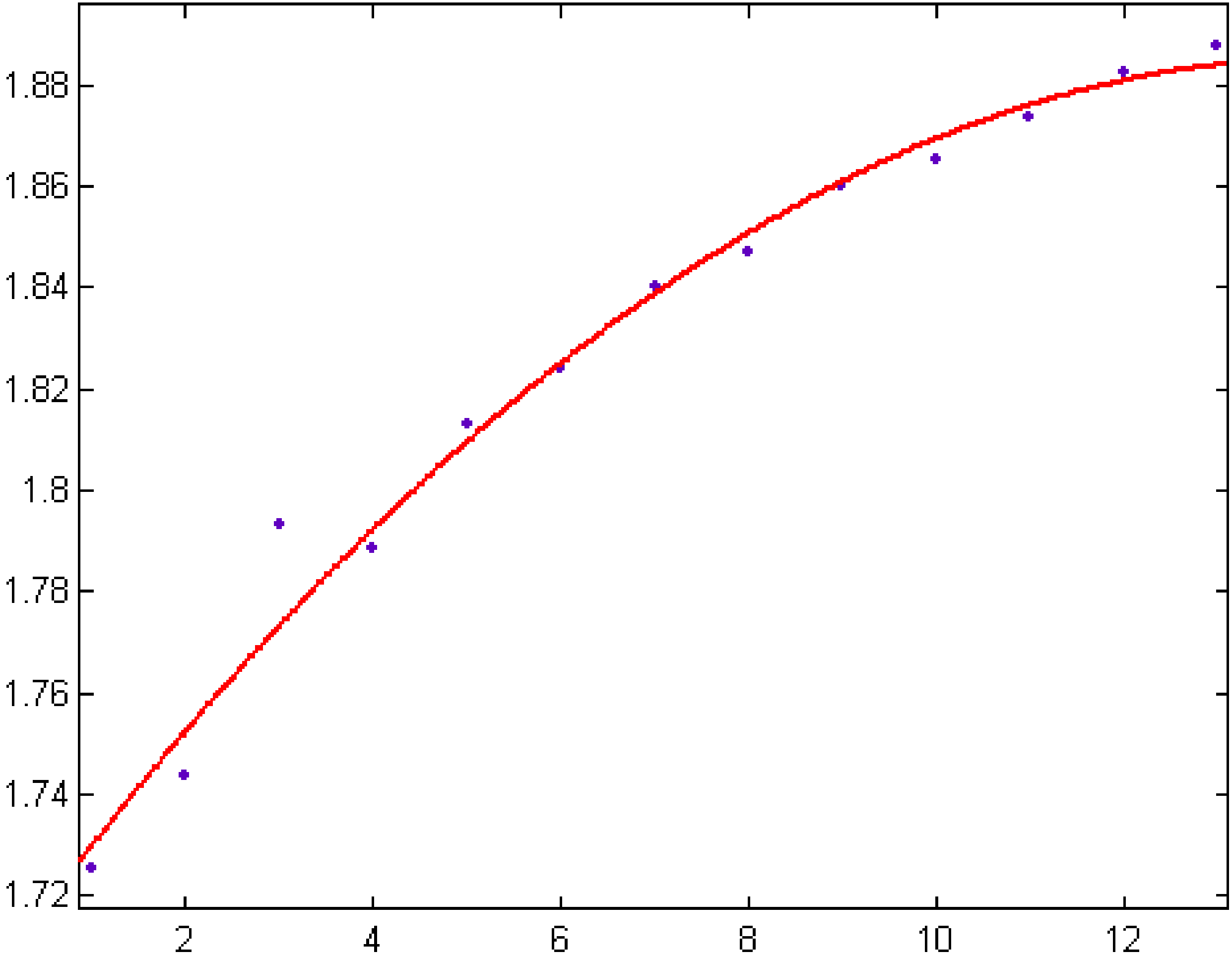


Figure 236
1.005
1
0.995
0.99
0.985
0
2
4
6
8
10
12
14

Figure 237



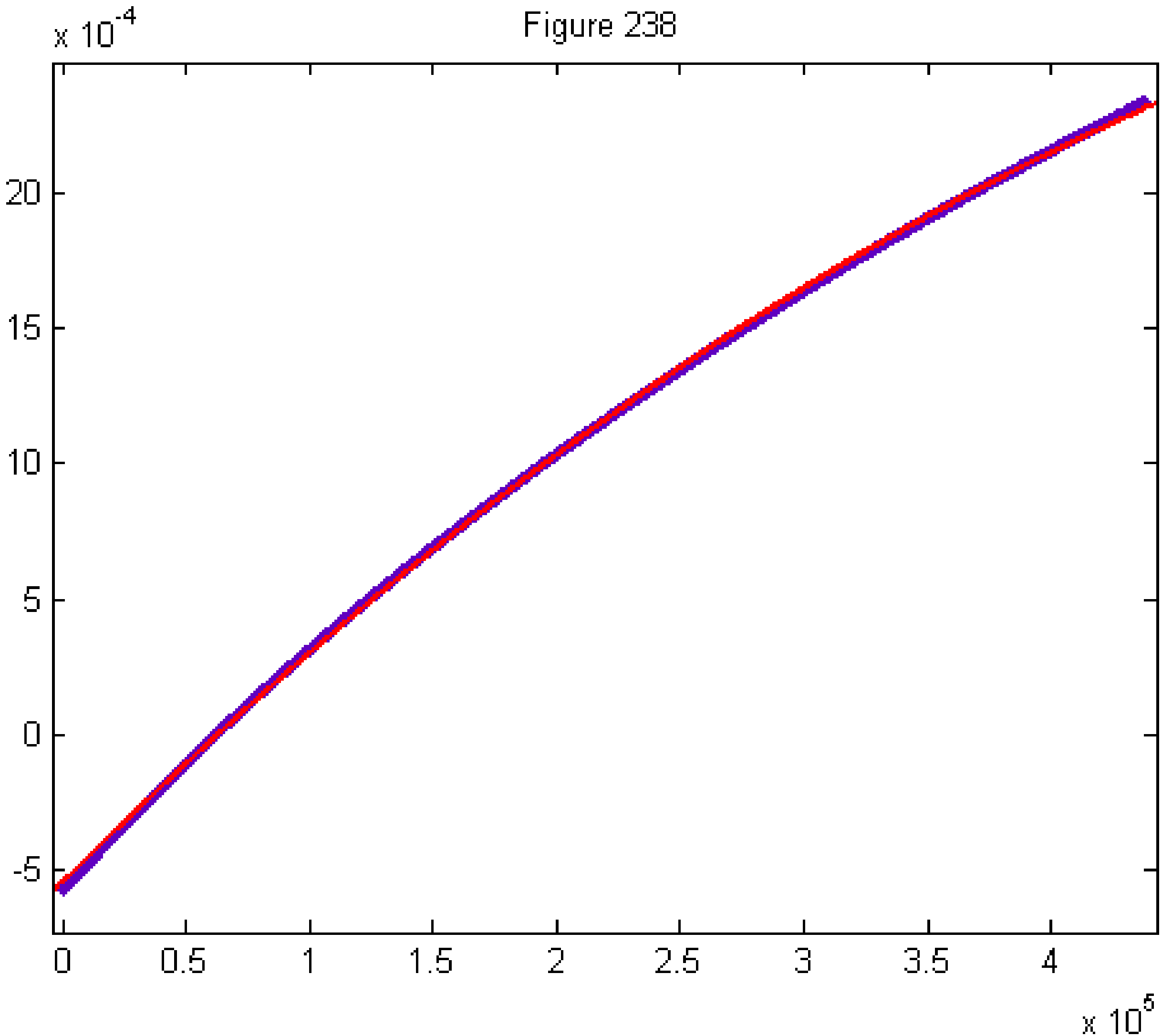
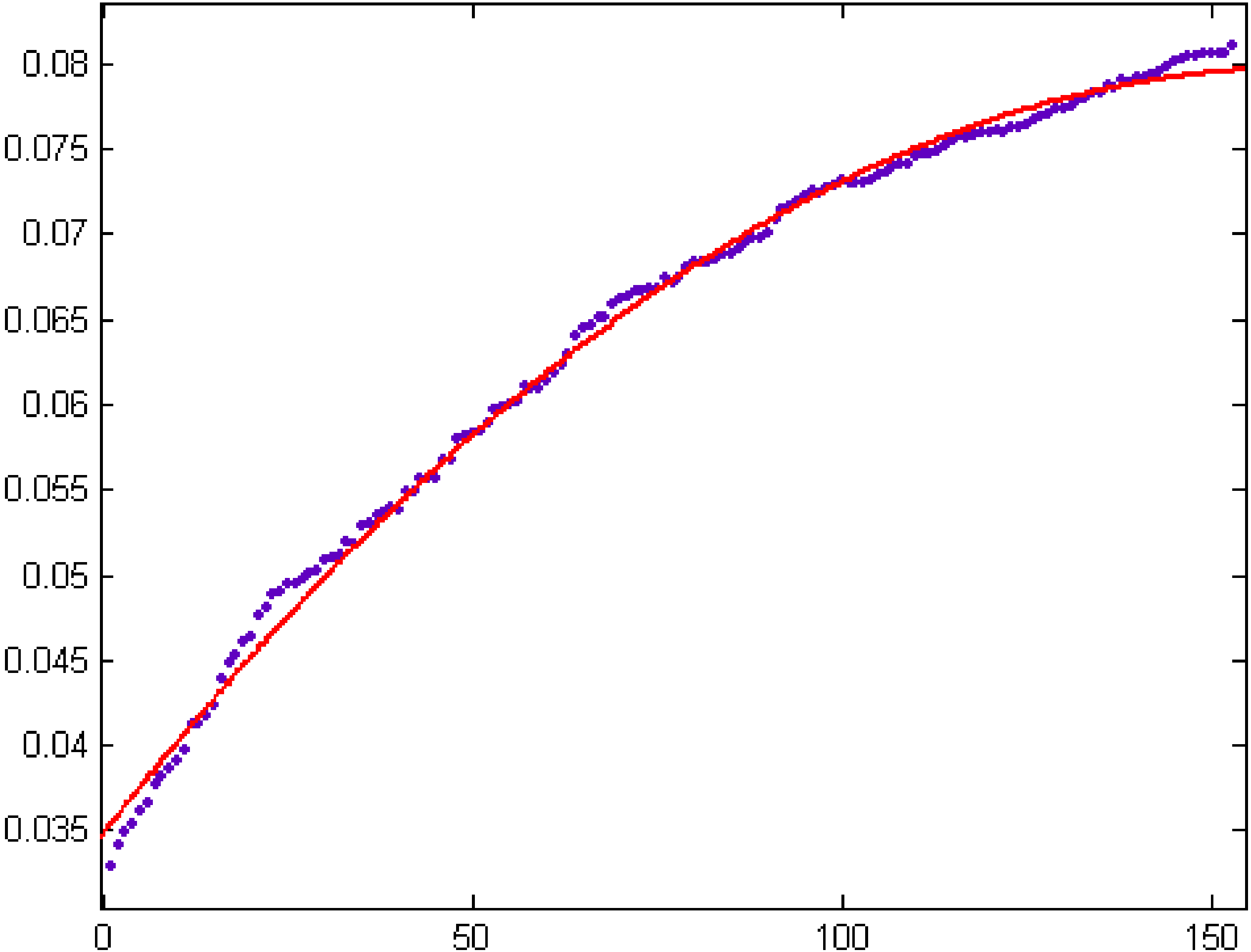


Figure 238
x 10^-4
20
15
10
5
0
-5
0
0.5
1
1.5
2
2.5
3
3.5
4
x 10^5

Figure 239



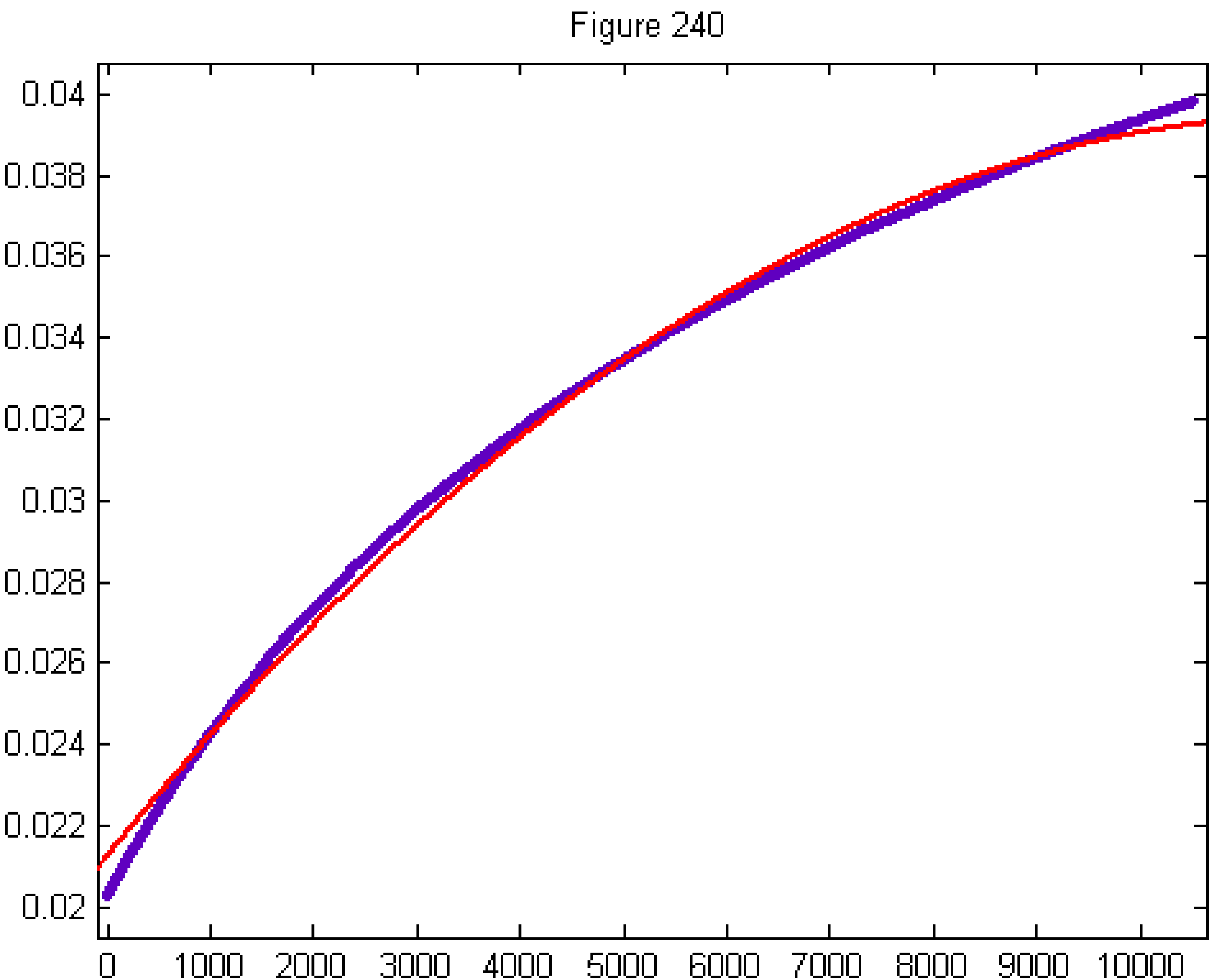


Figure 240
0.04
0.038
0.036
0.034
0.032
0.03
0.028
0.026
0.024
0.022
0.02
0
1000
2000
3000
4000
5000
6000
7000
8000
9000
10000

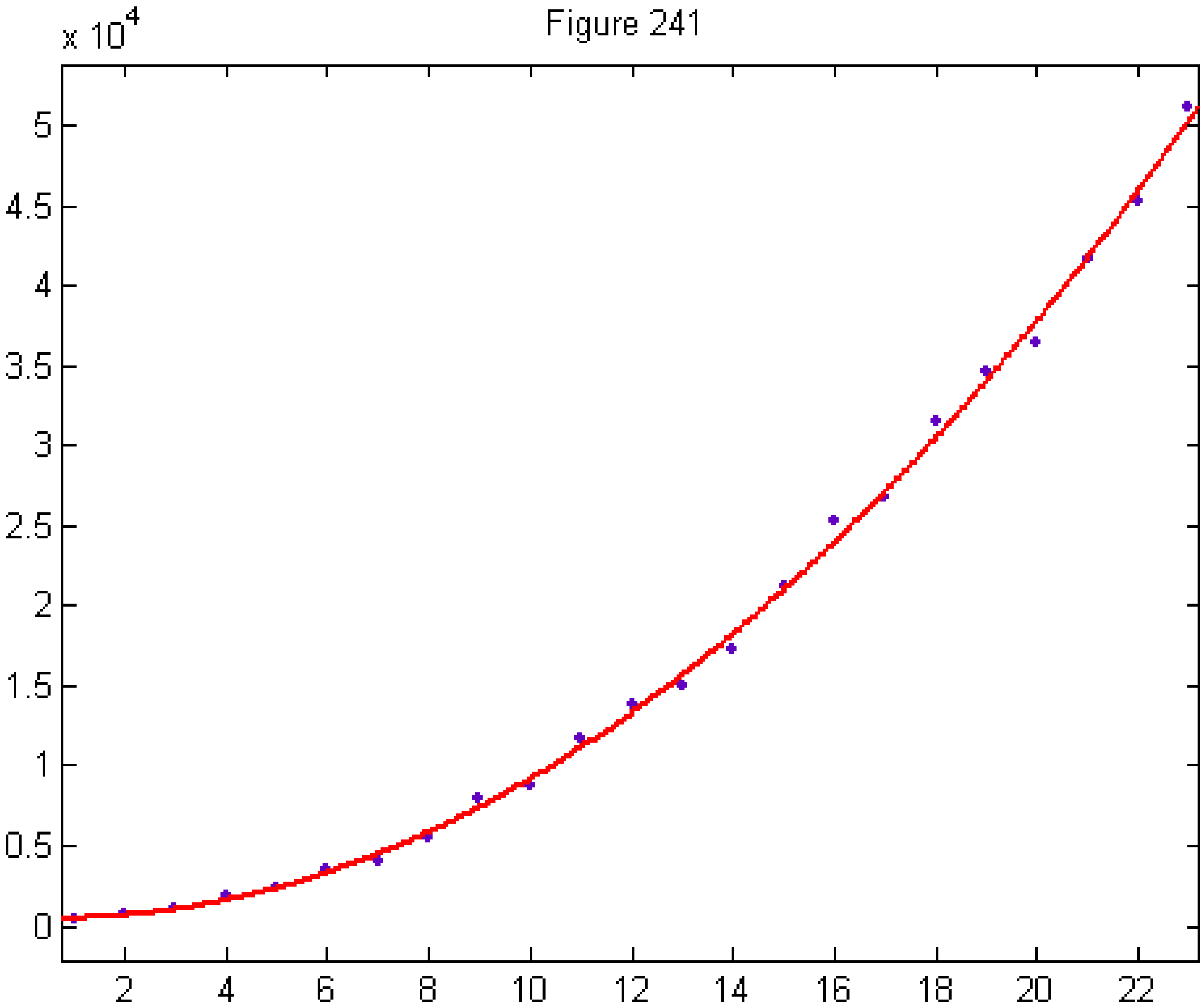
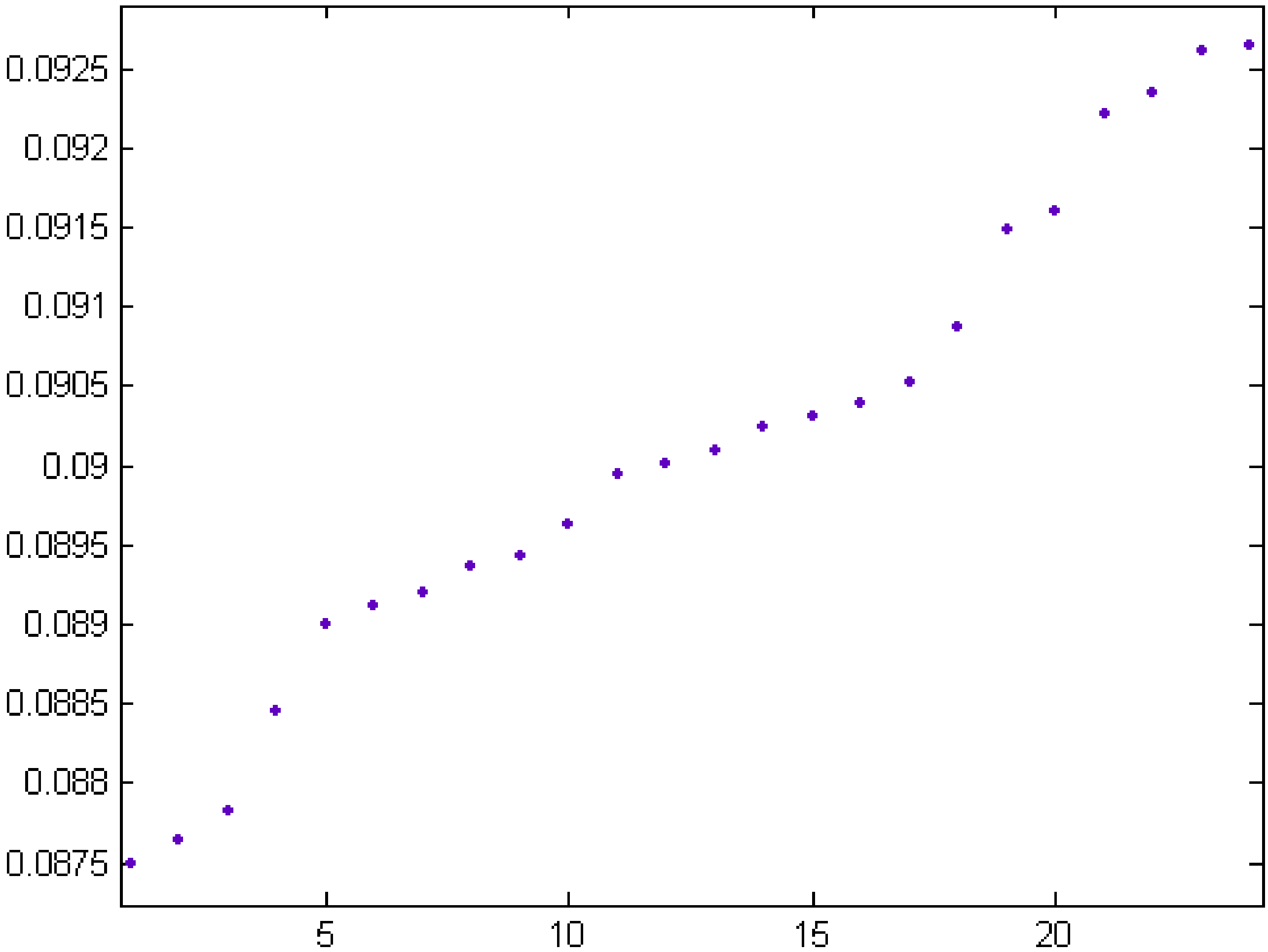


Figure 241
x 10^4
5
4.5
4
3.5
3
2.5
2
1.5
1
0.5
0
2
4
6
8
10
12
14
16
18
20
22

Figure 242



0.0925
0.092
0.0915
0.091
0.0905
0.09
0.0895
0.089
0.0885
0.088
0.0875
5
10
15
20

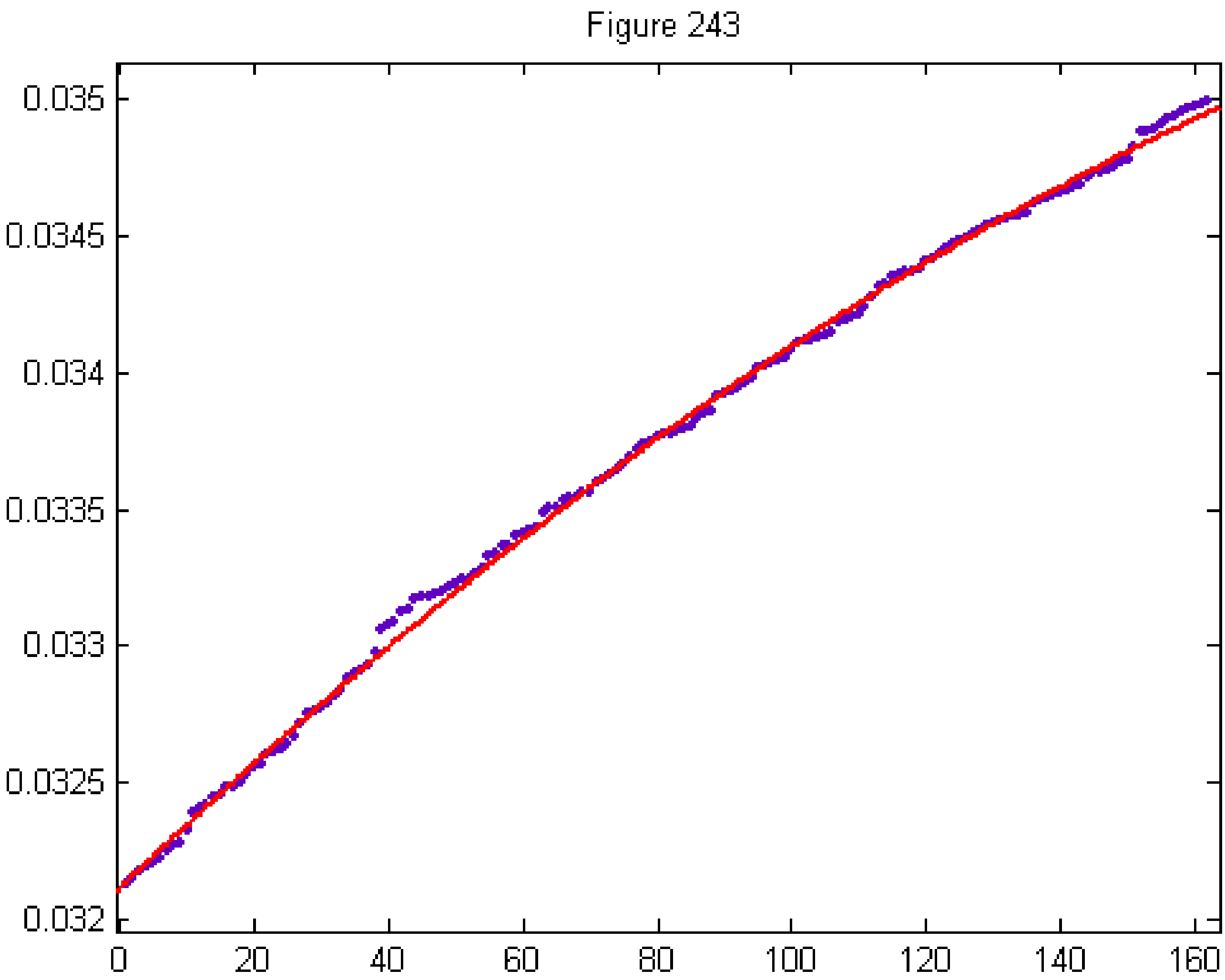


Figure 243
0.035
0.0345
0.034
0.0335
0.033
0.0325
0.032
0
20
40
60
80
100
120
140
160

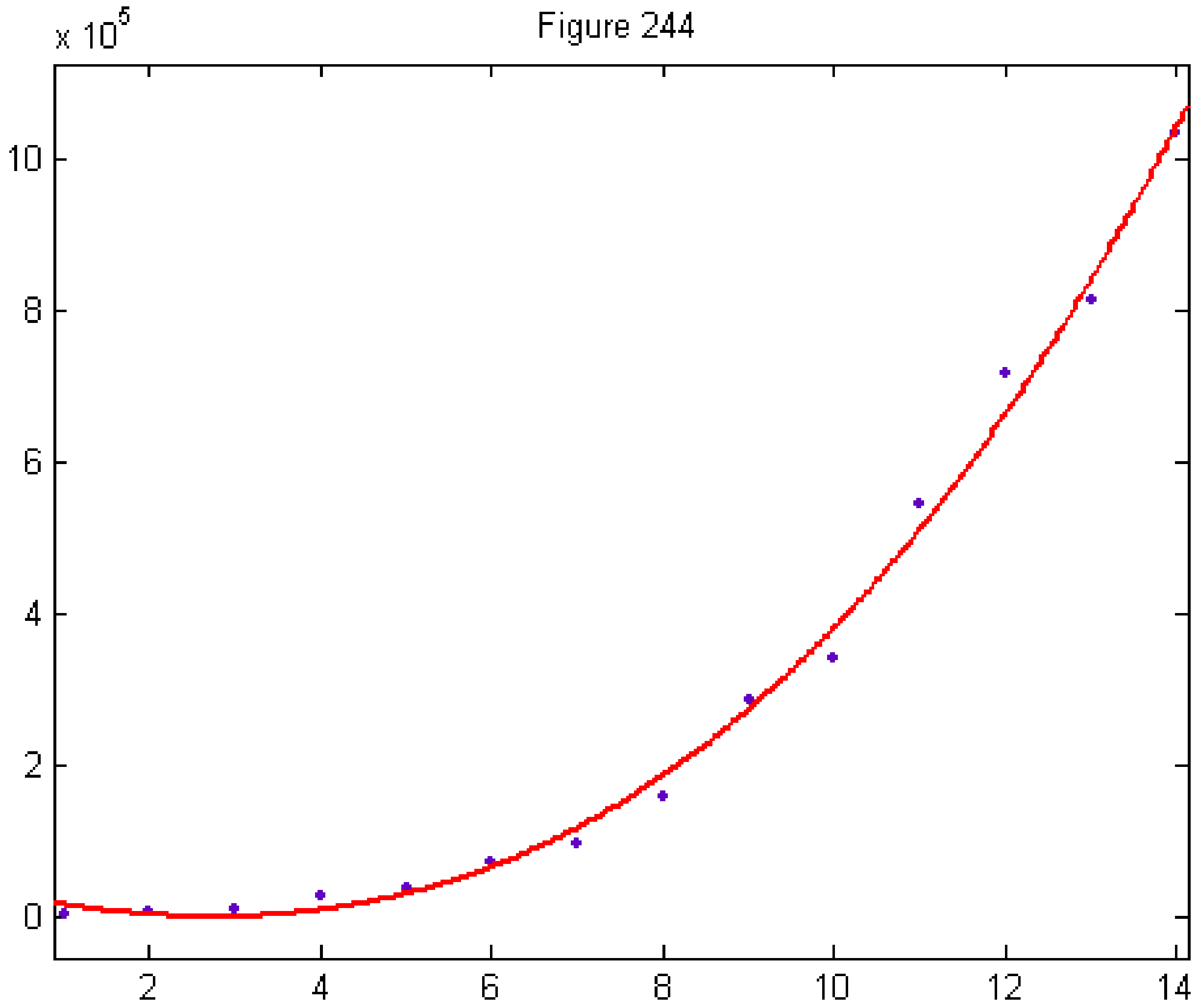


Figure 244
x 10^5
10
8
6
4
2
0
2
4
6
8
10
12
14

Figure 245

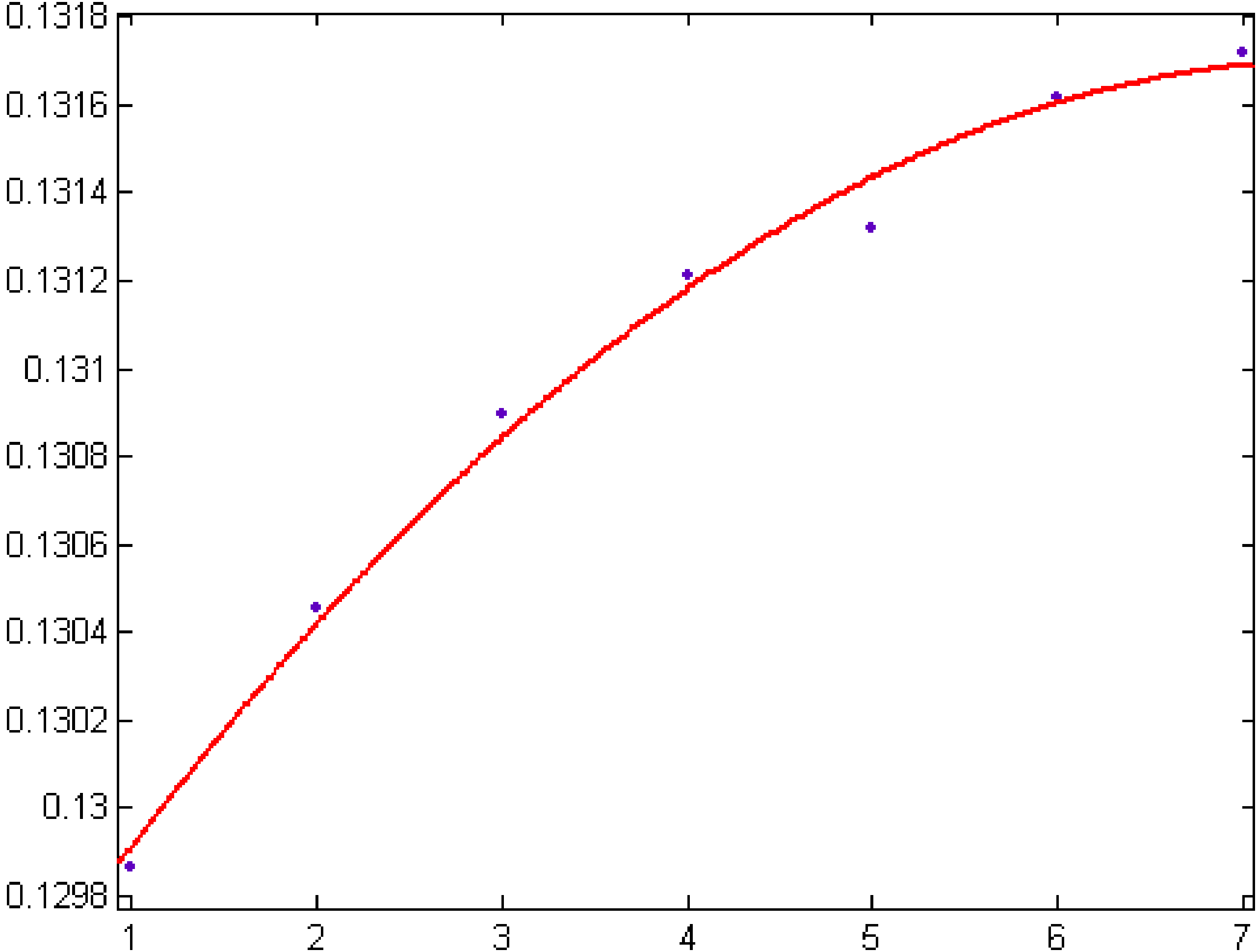
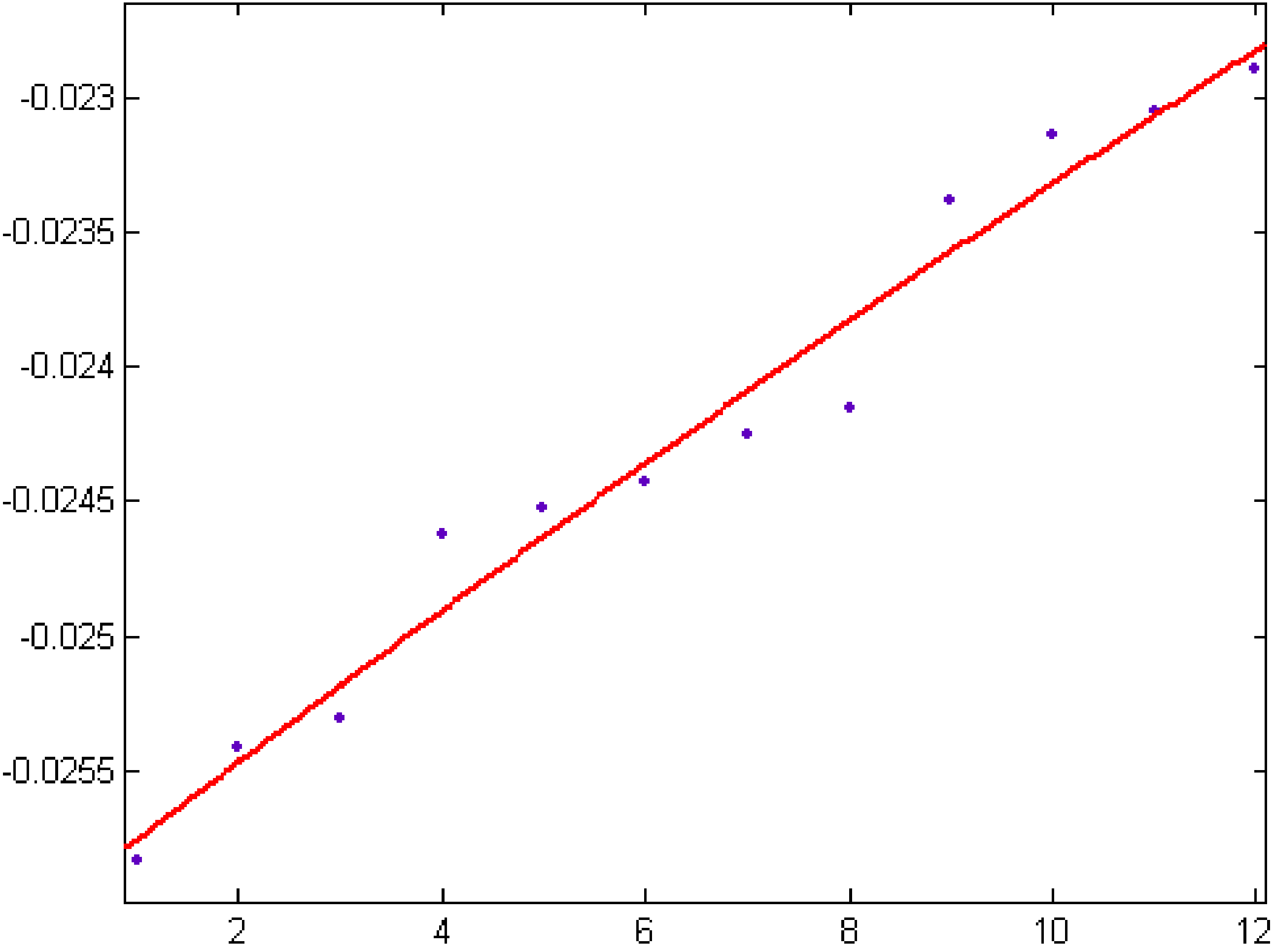

0.1318
0.1316
0.1314
0.1312
0.131
0.1308
0.1306
0.1304
0.1302
0.13
0.1298
1
2
3
4
5
6
7

Figure 246



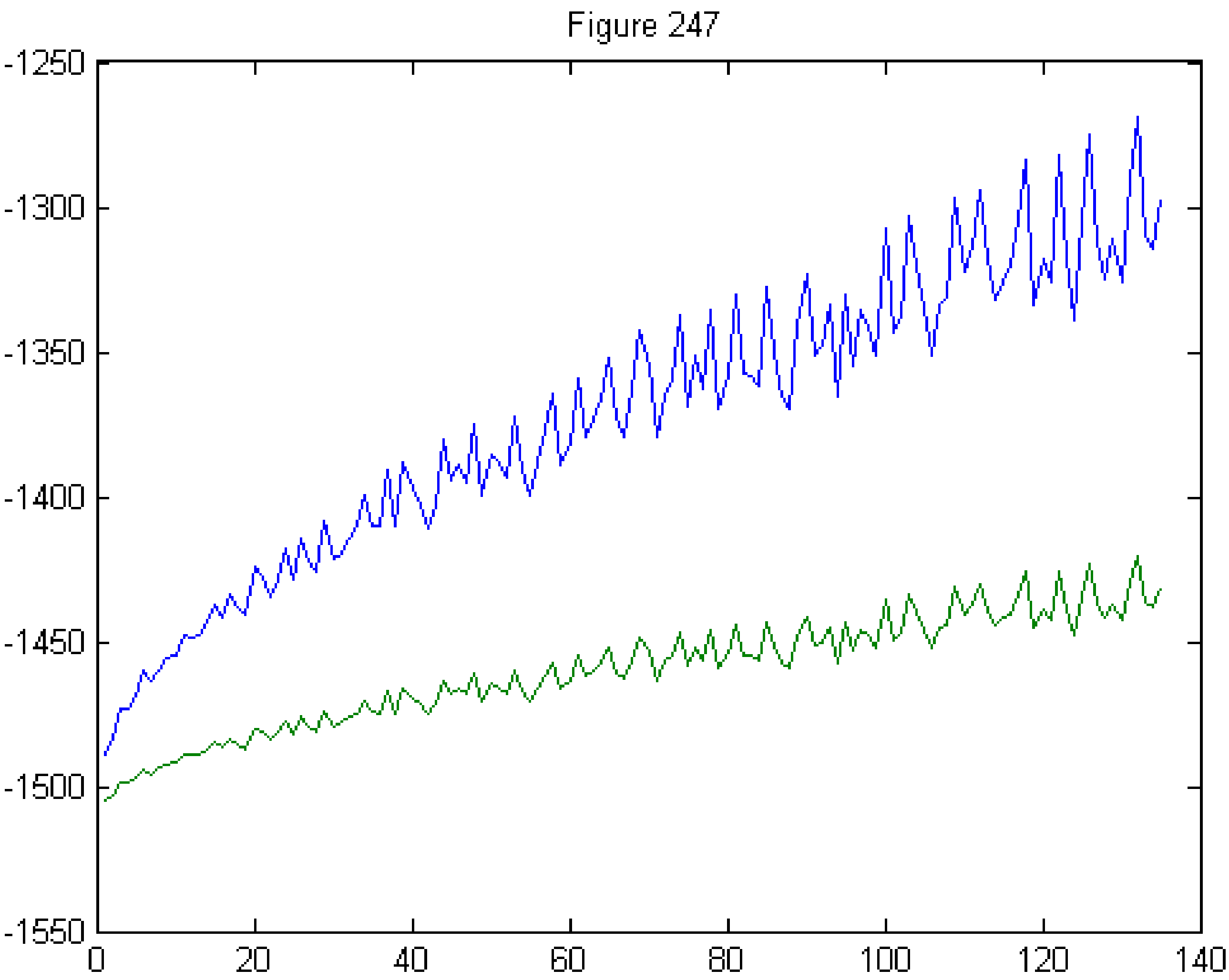
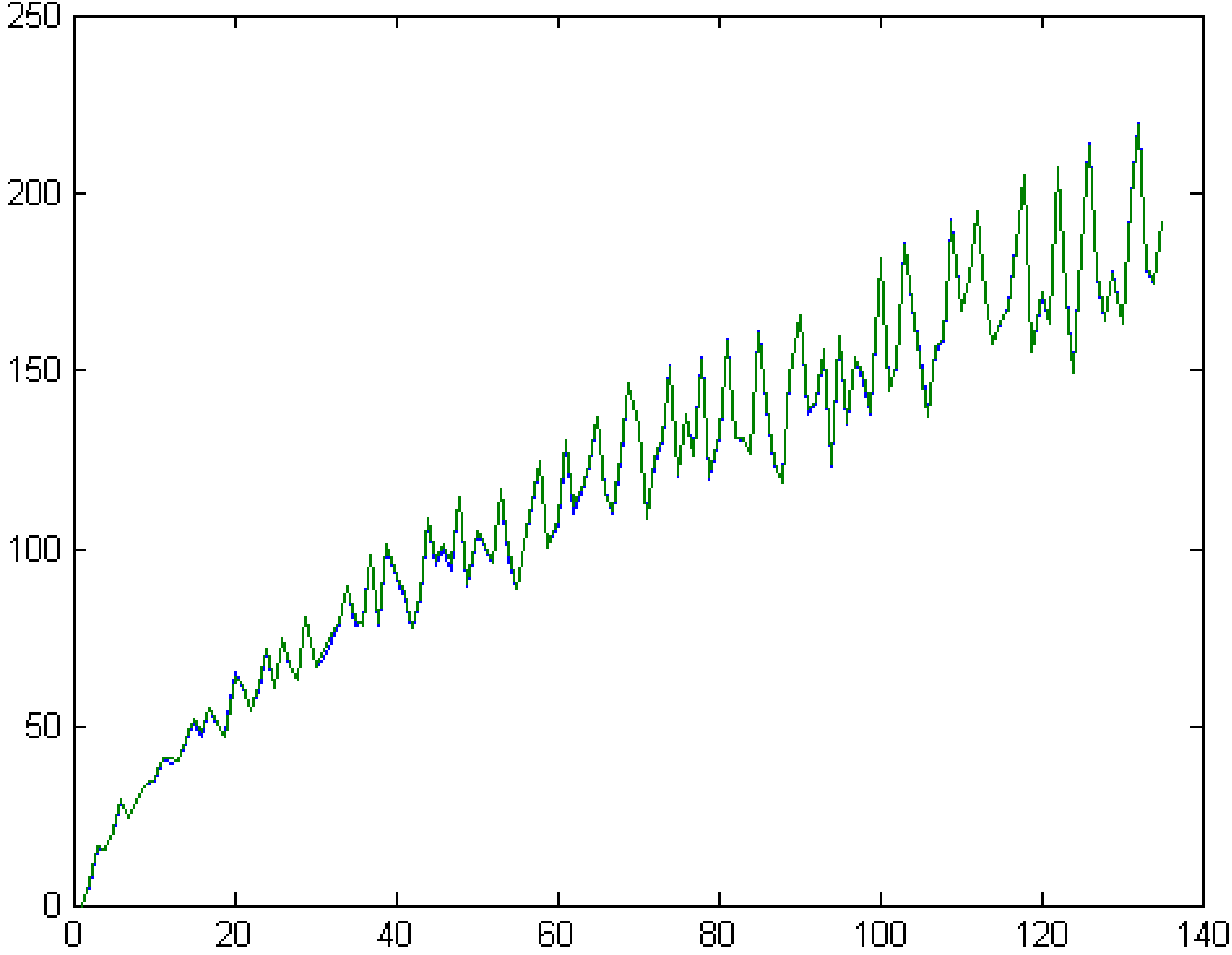


Figure 247
-1250
-1300
-1350
-1400
-1450
-1500
-1550
0
20
40
60
80
100
120
140

Figure 248



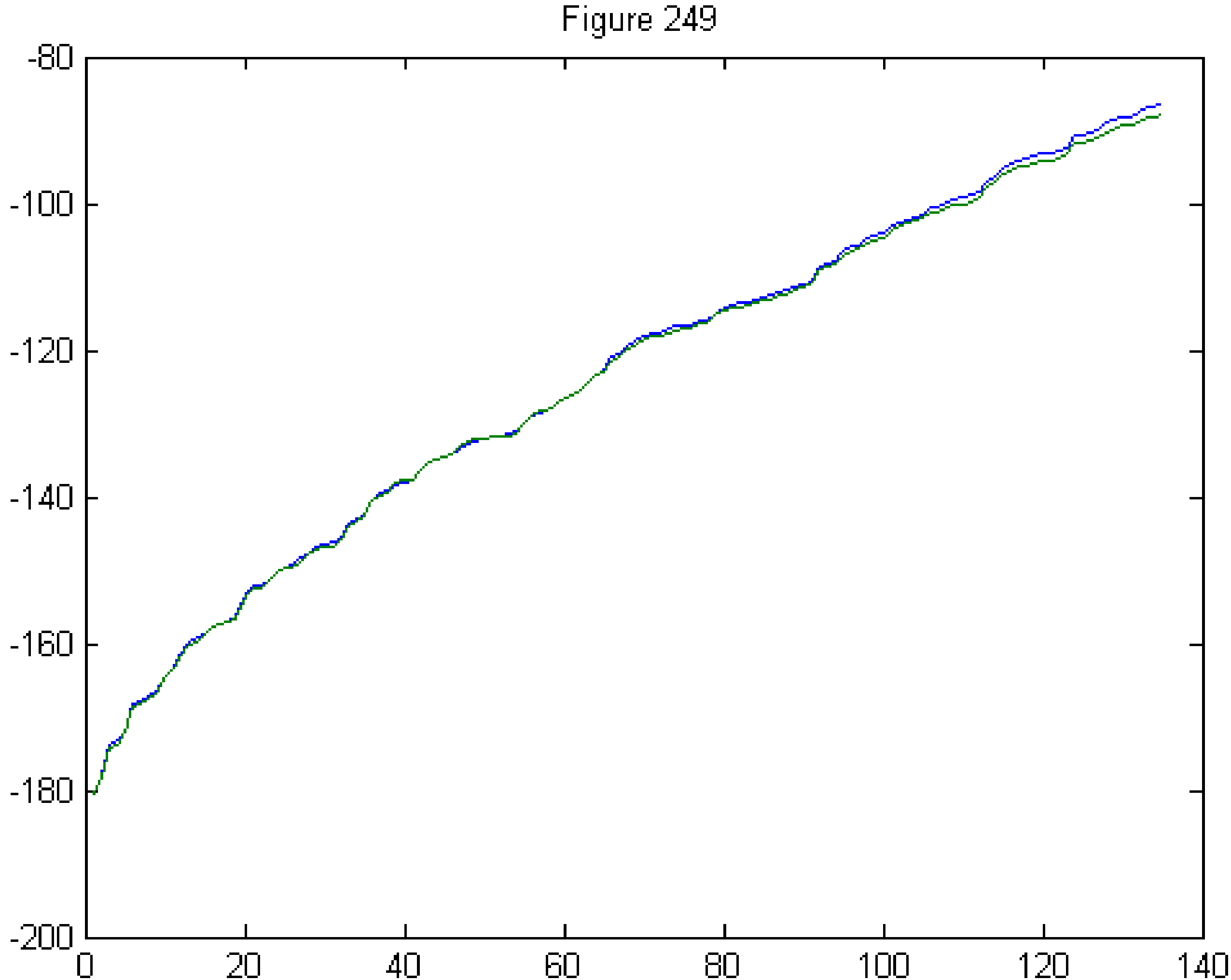
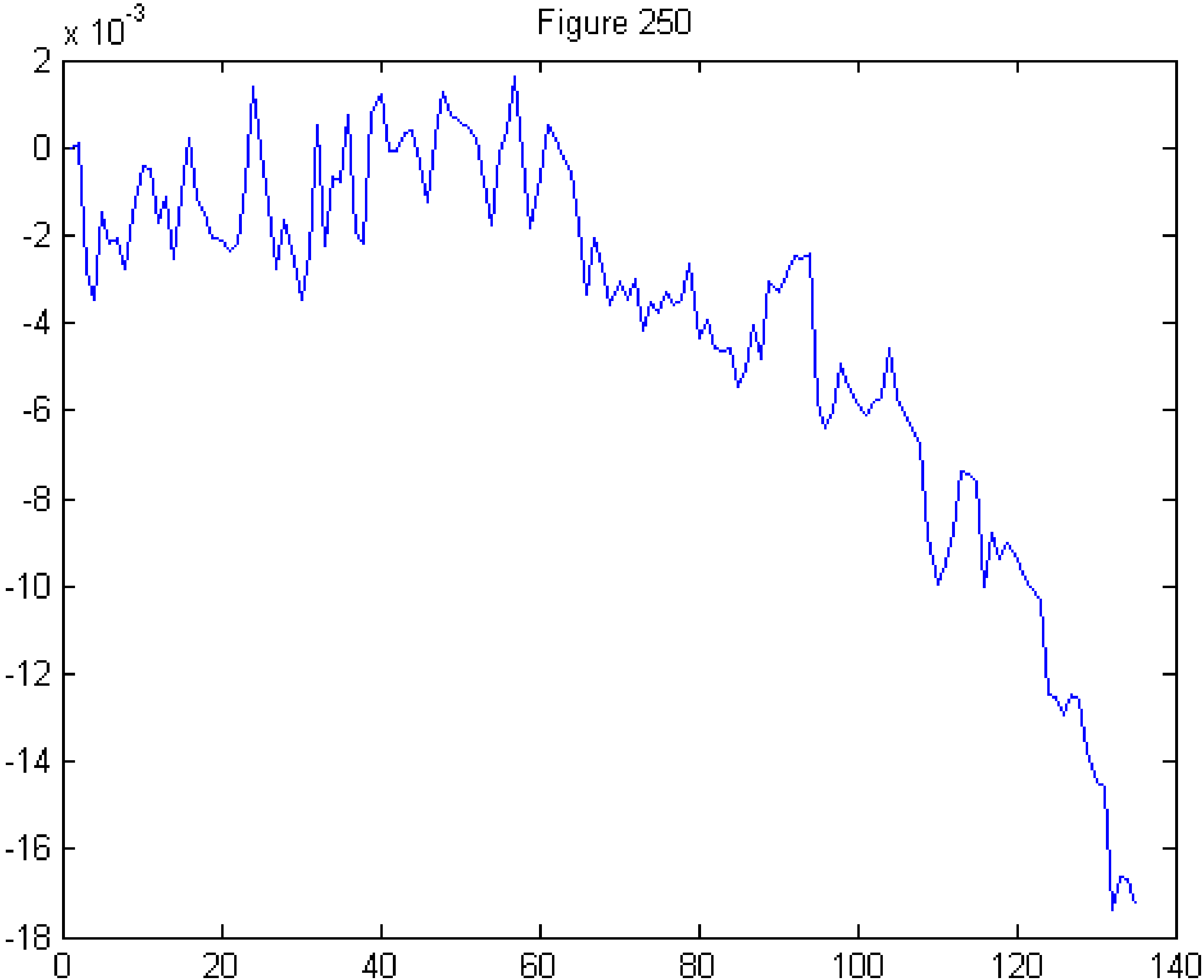


Figure 249
-80
-100
-120
-140
-160
-180
-200
0
20
40
60
80
100
120
140

Figure 250



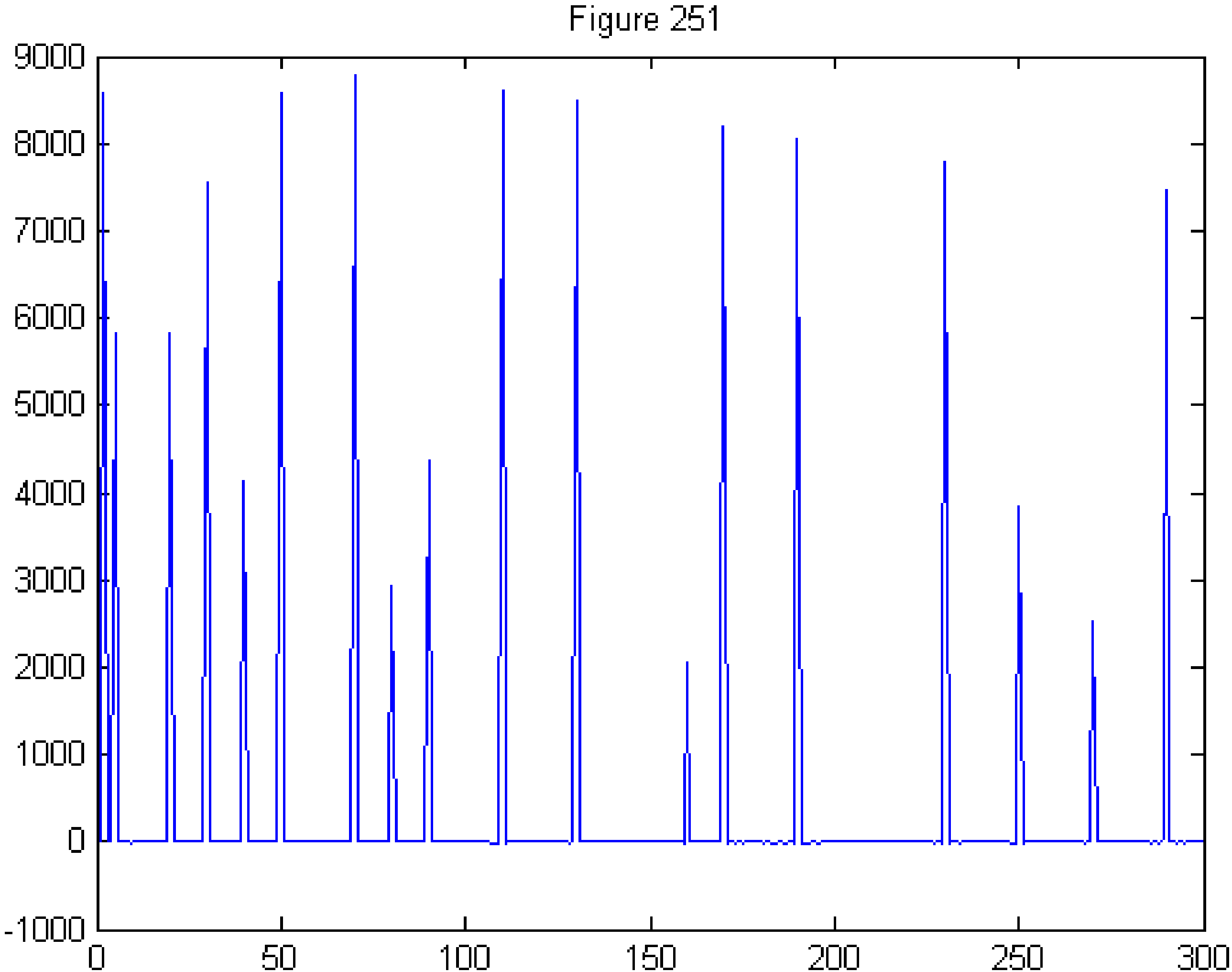


Figure 251
9000
8000
7000
6000
5000
4000
3000
2000
1000
0
-1000
0
50
100
150
200
250
300

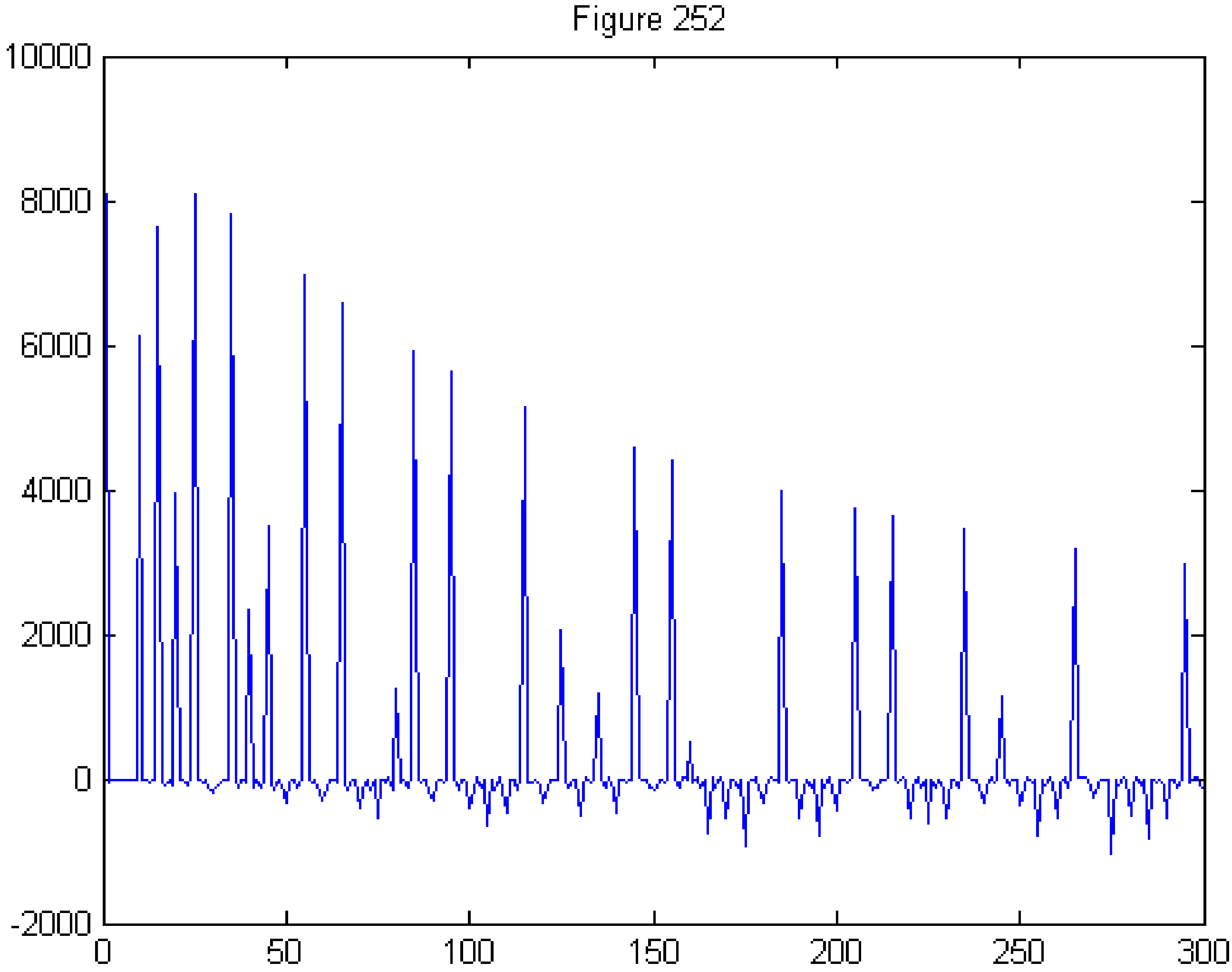


Figure 252
10000
8000
6000
4000
2000
0
-2000
0
50
100
150
200
250
300

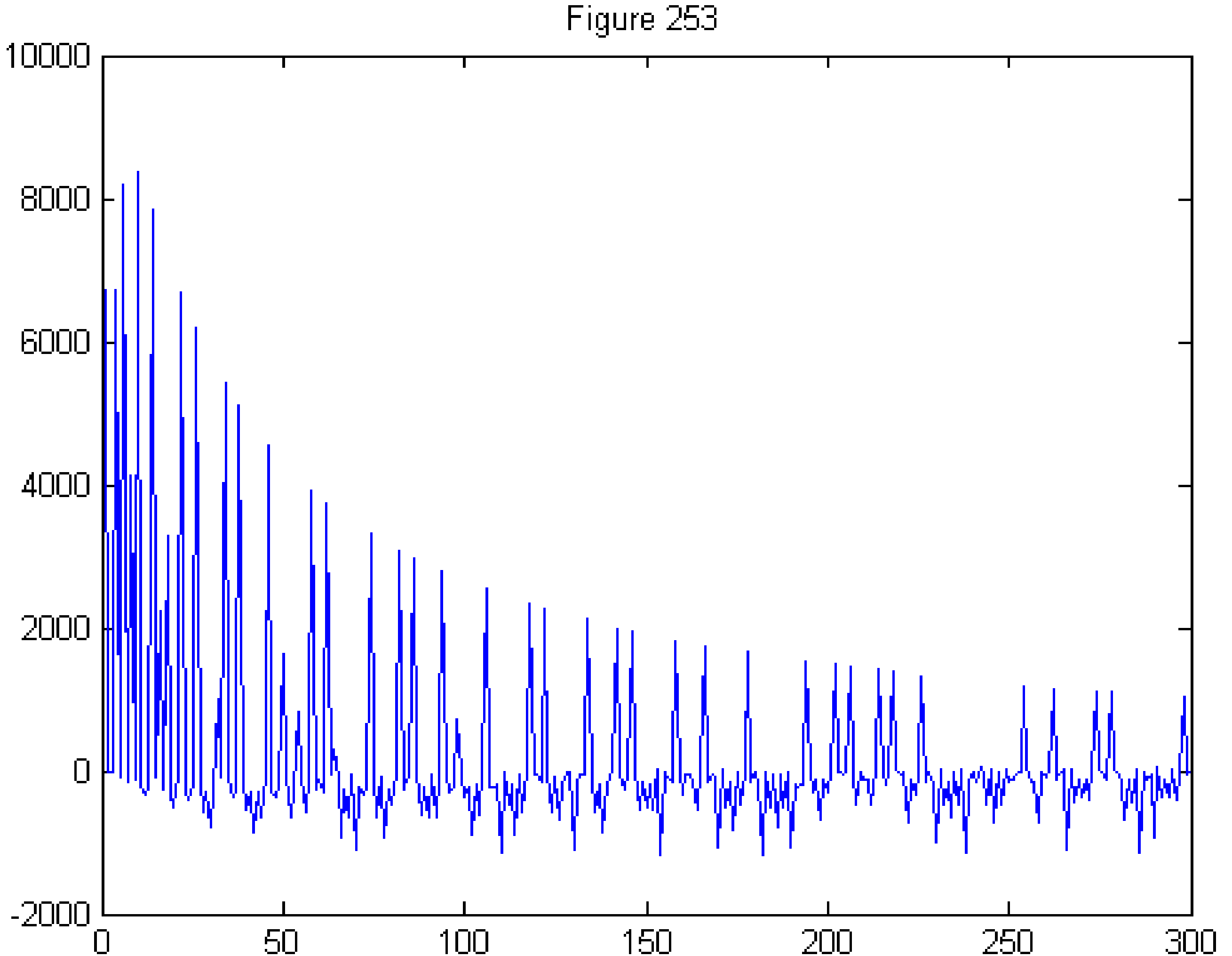
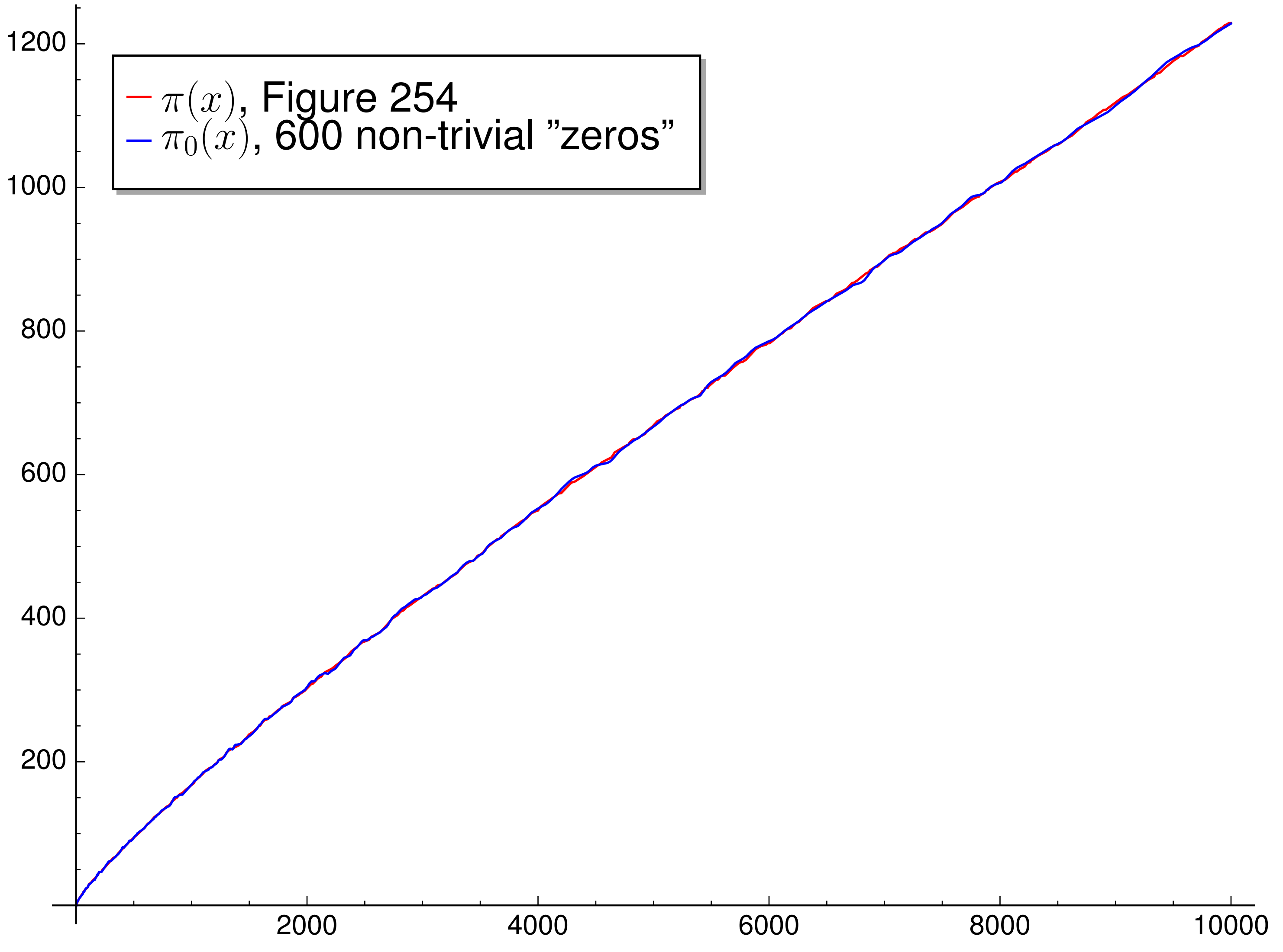


Figure 253
10000
8000
6000
4000
2000
0
-2000
0
50
100
150
200
250
300



$\pi(x)$, Figure 254
$\pi_0(x)$, 600 non-trivial "zeros"
1200
1000
800
600
400
200
2000
4000
6000
8000
10000